MEM 255 Introduction to Control Systems: *Solving State Equations*

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Outline

Goal: Introduce the concepts and terminology that underlay the state space tools implemented in MATLAB and similar software.

- **Problem definition**
- **Solving state space equations via Laplace transforms**
- **Solving state space equations in the time domain**
  - **Basic properties**
  - **The homogeneous equation/state transition matrix**
  - **Variation of parameters formula**

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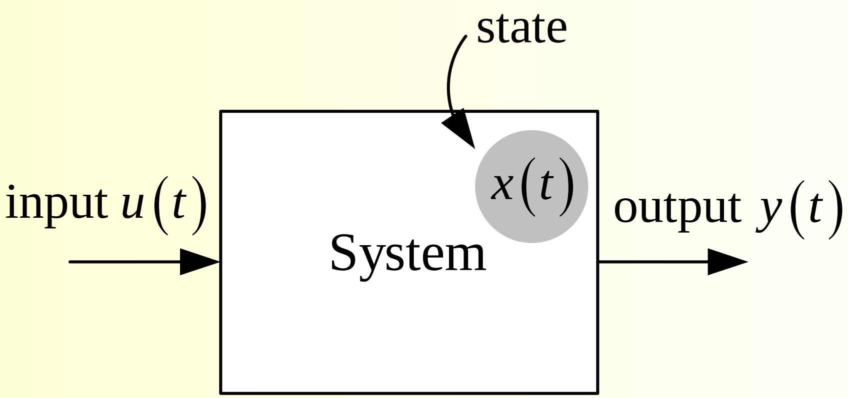
State Space Models



The diffrential equation or 'state space' model is

$\dot{x}(t) = Ax(t) + Bu(t)$ state equation

$y(t) = Cx(t) + Du(t)$ output equation

$x(0) = x_0$ initial condition

The state space model describes how the input $u(t)$ and the initial condition affect the state $x(t)$ and the output $y(t)$.

Solving Linear State Equations

$$\dot{x} = Ax + Bu(t), \quad x \in R^n, u \in R^m$$

$$given: x(t_0) = x_0, u(t) \ for \ t \geq t_0$$

$$find: x(t) \ for \ t \geq t_0$$

$\dot{x} = Ax + b(t), \quad b(t) := Bu(t),$ forced or nonhomogeneous

$\dot{x} = Ax$ homogeneous

Solving State Equations via the Laplace Transform

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$\begin{aligned} \mathscr{L}(\dot{x}) &= A\mathscr{L}(x) + B\mathscr{L}(u) \\ \mathscr{L}(y) &= C\mathscr{L}(x) + D\mathscr{L}(u) \end{aligned} \quad \Rightarrow \quad \begin{aligned} sX(s) - x_0 &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$X(s) = [sI - A]^{-1} x_0 + [sI - A]^{-1} BU(s)$$

$$Y(s) = C[sI - A]^{-1} x_0 + \left\{ C[sI - A]^{-1} B + D \right\} U(s)$$

The Resolvent

$$[sI - A]^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \approx \frac{n \times n \text{ matrix}}{\det(sI - A)}$$

$\text{adj}(sI - A) = n \times n$ matrix of cofactors

Recall, the n^2 minors of an $n \times n$ matrix M are defined as: the i, j minor M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained from M by deleting the i^{th} row and j^{th} column.

The i, j cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$

Basic properties

$$x_1(t),\, x_2(t) \quad \text{sol'ns of homog.}, c_1, c_2 \text{ constants}$$

$$\Downarrow$$

$$x(t) = c_1 x_1(t) + c_2 x_2(t) \text{ is a sol'n of homog.}$$

$$x_1(t),\, x_2(t) \quad \text{sol'ns of forced}$$

$$\Downarrow$$

$$x(t) = x_1(t) - x_2(t) \text{ is a sol'n of homog.}$$

$$x_p(t) \text{ any sol'n of forced}, x_h(t) \text{ any sol'n of homog.}$$

$$\Downarrow$$

$$x(t) = x_h(t) + x_p(t) \text{ is a sol'n of forced}$$

The Homogeneous Equation

Let us first solve the homogeneous equation

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0$$

Strategy: assume a sol'n and see if it works.

Assume a solution in the form of a power series:

$$x(t) = \mathbf{a}_0 + \mathbf{a}_1(t - t_0) + \mathbf{a}_2(t - t_0)^2 + \cdots + \mathbf{a}_k(t - t_0)^k + \cdots$$

$$\dot{x}(t) \Rightarrow \mathbf{a}_1 + 2\mathbf{a}_2(t - t_0) + \cdots + k\mathbf{a}_k(t - t_0)^{k-1} + \cdots$$

$$Ax(t) \Rightarrow A\mathbf{a}_0 + A\mathbf{a}_1(t - t_0) + A\mathbf{a}_2(t - t_0)^2 + \cdots + A\mathbf{a}_k(t - t_0)^k + \cdots$$

Compare coefficients of like powers of $(t - t_0)$

The Homogeneous Equations, 2

$$\begin{aligned}
\mathbf{a}_1 &= A\mathbf{a}_0 & \qquad & \mathbf{a}_1 = A\mathbf{a}_0 \\
\mathbf{a}_2 &= \frac{1}{2}A\mathbf{a}_1 & & \mathbf{a}_2 = \frac{1}{2}A^2\mathbf{a}_0 \\
&\vdots & \Rightarrow & \quad \vdots \\
\mathbf{a}_k &= \frac{1}{k}A\mathbf{a}_{k-1} & & \mathbf{a}_k = \frac{1}{k!}A^k\mathbf{a}_0 \\
&\vdots & & \quad \vdots
\end{aligned}$$

$$\begin{aligned}
x(t) &= \mathbf{a}_0 + A\mathbf{a}_0(t-t_0) + \frac{1}{2}A^2\mathbf{a}_0(t-t_0)^2 + \cdots + \frac{1}{k!}A^k\mathbf{a}_0(t-t_0)^k + \cdots \\
&= \left[I + A(t-t_0) + \frac{1}{2}A^2(t-t_0)^2 + \cdots + \frac{1}{k!}A^k(t-t_0)^k + \cdots\right]\mathbf{a}_0
\end{aligned}$$

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The Homogeneous Equation, 3

Set $t = t_0,\ x(t_0) = x_0$ to obtain

$$\mathbf{a}_0 = x_0 \Rightarrow$$

$$x(t) = \left[I + A(t-t_0) + \frac{1}{2}A^2(t-t_0)^2 + \cdots + \frac{1}{k!}A^k(t-t_0)^k + \cdots \right] x_0$$

Recall the series expansion for the (scalar) exponential

$$e^{a(t-t_0)} = 1 + a(t-t_0) + \frac{1}{2}a^2(t-t_0)^2 + \cdots + \frac{1}{k!}a^k(t-t_0)^k + \cdots$$

Define the matrix exponential

$$e^{A(t-t_0)} \triangleq I + A(t-t_0) + \frac{1}{2}A^2(t-t_0)^2 + \cdots + \frac{1}{k!}A^k(t-t_0)^k + \cdots$$

so that

State transition matrix

$$x(t) = e^{A(t-t_0)}x_0 = \Phi(t,t_0)\,x_0$$

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Matrix Exponential

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

Some properties:

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

$$e^{At}e^{-At} = I \Rightarrow \left[e^{At}\right]^{-1} = e^{-At}$$

$$e^{At}e^{Bt} = e^{(A+B)t} \text{ if and only if } AB = BA$$

Variation of Parameters Formula

Recall, any sol'n of (forced) satisfies

$$x(t) = x_h(t) + x_p(t)$$

where

$x_h(t) = e^{At}c$ for constant vector c, satisfies (homog.)

$x_p(t)$ is any (particular) sol'n of (forced)

We seek $x_p(t)$.

Assume the form $x_p(t) = e^{At}c(t)$.

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Variation of Parameters, 2

$$\dot{x}_p = Ax_p + Bu \text{ and } \frac{d}{dt}e^{At}c(t) = Ae^{At}c(t) + e^{At}\dot{c}(t)$$

$$\Rightarrow \dot{c}(t) = e^{-At}B(t)u(t)$$

$$\Rightarrow c(t) = \cancel{c(t_0)} + \int_{t_0}^{t} e^{-A\tau}B(\tau)u(\tau)\,d\tau$$

Now,

$$x(t) = e^{At}c + e^{At}\int_{t_0}^{t} e^{-A\tau}B(\tau)u(\tau)\,d\tau$$

$$= e^{At}c + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)\,d\tau$$

Variation of Parameters, 3

Finally, $x(t_0) = x_0 \Rightarrow x_0 = e^{At_0} c \Rightarrow c = e^{-At_0} x_0$

$$\begin{aligned} x(t) &= e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B(\tau) u(\tau)\, d\tau \\ &= \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau)\, d\tau \end{aligned}$$

Recall (with t_0=0)

$$X(s) = [sI - A]^{-1} x_0 + [sI - A]^{-1} BU(s)$$

By comparison,

$$\mathscr{L}\left[e^{At}\right] = [sI - A]^{-1}$$

Example - Mathematica

In[1]:= $\text{MatrixExp}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} t\right]$ // MatrixForm

ut[1]//MatrixForm=

$$\begin{pmatrix} e^t & 0 & 0 \\ (-1+e^t)^2 & e^{2t} & \frac{1}{2}(-1+e^{2t}) \\ 2(-1+e^t) & 0 & 1 \end{pmatrix}$$

In[4]:= $\text{LaplaceTransform}\left[\text{MatrixExp}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} t\right], t, s\right]$ // MatrixForm

ut[4]//MatrixForm=

$$\begin{pmatrix} \frac{1}{-1+s} & 0 & 0 \\ \frac{1}{-2+s} - \frac{2}{-1+s} + \frac{1}{s} & \frac{1}{-2+s} & \frac{1}{2}\left(\frac{1}{-2+s} - \frac{1}{s}\right) \\ 2\left(\frac{1}{-1+s} - \frac{1}{s}\right) & 0 & \frac{1}{s} \end{pmatrix}$$

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Example - MATLAB

```
>> A=[1 0 0;0 2 1;2 0 0];
>> syms t
>> expm(t*A)

[              exp(t),                    0,                    0]
[ exp(2*t)-2*exp(t)+1,             exp(2*t),   -1/2+1/2*exp(2*t)]
[          2*exp(t)-2,                    0,                    1]
>> laplace(expm(t*A))

[             1/(s-1),                    0,                    0]
[ 1/(s-2)-2/(s-1)+1/s,              1/(s-2),    -1/2/s+1/2/(s-2)]
[         2/(s-1)-2/s,                    0,                  1/s]
```

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Summary

- State transition matrix
- Matrix exponential
- Resolvent
- Variation of parameters formula

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