MEM 255 Introduction to Control Systems: *Modes-eigenvalues & eigenvectors*

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Outline

- Similarity Transformations
- Eigenvalues & Eigenvectors
- Using MATLAB
- Diagonal Form
- Modes
- Complex eigenvalues

Similarity Transformations

$$\dot{x} = Ax + Bu \qquad \dot{x} = Ax + bu$$
$$y = Cx + Du \qquad x \in R^n, u \in R^m, y \in R^p \qquad y = cx + du$$

Now consider the transformation to new states *z*, defined by

$$x = Tz \Leftrightarrow z = T^{-1}x$$

$$\begin{aligned} T\dot{z} &= ATz + Bu \\ y &= CTz + Du \end{aligned} \Rightarrow \begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned}$$

so that,

$$\begin{aligned} \dot{z} &= A^* z + B^* u \\ y &= C^* z + D^* u \end{aligned}, \quad A^* = T^{-1}AT, B^* = T^{-1}B, C^* = CT, D^* = D$$

Eigenvalues & Eigenvectors

Consider the square $n \times n$ matrix A as a map from R^n to R^n, i.e.,

$$y = Ax$$

Does there exist a nontrivial input vector h, such that the output vector y, points in the same direction as h, i.e., $y = \lambda h$, λ where is some scalar?

$$Ah = \lambda h$$

Let's try to solve for h

$[\lambda I - A]h = 0 \Rightarrow$ A nontrivial solution exists iff $\det[\lambda I - A] = 0$.

Characteristic polynomial $\phi(\lambda) \triangleq \det[\lambda I - A] = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$

$\phi(\lambda) = 0$ has n roots $\{\lambda_1, \ldots, \lambda_n\}$, possibly complex, possibly repeated.

Choose one, say λ_i, $i \in \{\lambda_1, \ldots, \lambda_n\}$ and find the corresponding h_i

$$[\lambda_i I - A]h_i = 0$$

Example 1 (distinct roots)

$$A=\begin{bmatrix}-5 & 3\\ 3 & -5\end{bmatrix}\Rightarrow \det\left[\lambda I-A\right]=\det\begin{bmatrix}\lambda+5 & -3\\ -3 & \lambda+5\end{bmatrix}=\left(\lambda+5\right)^2-9$$

$$\phi\left(\lambda\right)=\lambda^2+10\lambda+16=\left(\lambda+2\right)\left(\lambda+8\right)\Rightarrow \boxed{\lambda_1=-2,\lambda_2=-8}$$

$$\left[-2I-A\right]h_1=0\Rightarrow\begin{bmatrix}-2+5 & -3\\ -3 & -2+5\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=0$$

$$\begin{matrix}3\eta_1-3\eta_2=0\\ -3\eta_1+3\eta_2=0\end{matrix}\Rightarrow \eta_1=\eta_2\Rightarrow \boxed{h_1=\begin{bmatrix}1\\ 1\end{bmatrix}\eta_2}$$

$$\left[-8I-A\right]h_1\Rightarrow\begin{matrix}-3\eta_1-3\eta_2=0\\ -3\eta_1-3\eta_2=0\end{matrix}\Rightarrow \eta_1=-\eta_2\Rightarrow \boxed{h_2=\begin{bmatrix}-1\\ 1\end{bmatrix}\eta_2}$$

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Example 2 (repeated roots)

$$A=\begin{bmatrix}-5 & 0\\ 0 & -5\end{bmatrix} \Rightarrow \det[\lambda I - A] = \det\begin{bmatrix}\lambda+5 & 0\\ 0 & \lambda+5\end{bmatrix} = (\lambda+5)^2$$

$$\phi(\lambda) = (\lambda+5)(\lambda+5) \Rightarrow \boxed{\lambda_1 = -5, \lambda_2 = -5}$$

$$[-5I - A]h_1 = 0 \Rightarrow \begin{bmatrix}-5+5 & 0\\ 0 & -5+5\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = 0 \text{ satisfied for any } \eta_1, \eta_2$$

$$h_{1,2} = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} \quad \text{Choose} \quad \boxed{h_1 = \begin{bmatrix}1\\ 0\end{bmatrix}, h_2 = \begin{bmatrix}0\\ 1\end{bmatrix}}$$

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Diagonal Form

eigen-system of A: $\begin{matrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \leftarrow \text{eigenvalues} \\ h_1 & h_2 & \cdots & h_n \leftarrow \text{independent eigenvectors} \end{matrix}$

$$T \triangleq \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}$$

$$\Rightarrow A^* = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} A \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} \begin{bmatrix} Ah_1 & Ah_2 & \cdots & Ah_n \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 h_1 & \lambda_2 h_2 & \cdots & \lambda_3 h_n \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{-1} \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$= diag\left(\lambda_1, \ldots \lambda_n\right)$$

$$\boxed{\dot{z}_i = \lambda_i z_i + b_i^* u,\ i = 1, \ldots, n}$$

A decoupled system of n 1st order ode's

Example

Define A

Compute eigensystem

Check similarity trans

Use linear solve rather than inv

```
>> A=[3 2 1;4 5 6;1 2 3];
>> [V,D]=eig(A)
V =
   -0.3482   -0.8581    0.4082
   -0.8704    0.1907   -0.8165
   -0.3482    0.4767    0.4082
D =

    9.0000         0         0
         0    2.0000         0
         0         0   -0.0000
>> inv(V)*A*V
ans =
    9.0000   -0.0000   -0.0000
   -0.0000    2.0000   -0.0000
   -0.0000   -0.0000   -0.0000
>> V\A*V
ans =
    9.0000   -0.0000    0.0000
   -0.0000    2.0000         0
   -0.0000   -0.0000    0.0000
```

Modal Coordinates, 1

Consider the system in diagonal form. The z coordinates are referred to as 'modal coordinates'.

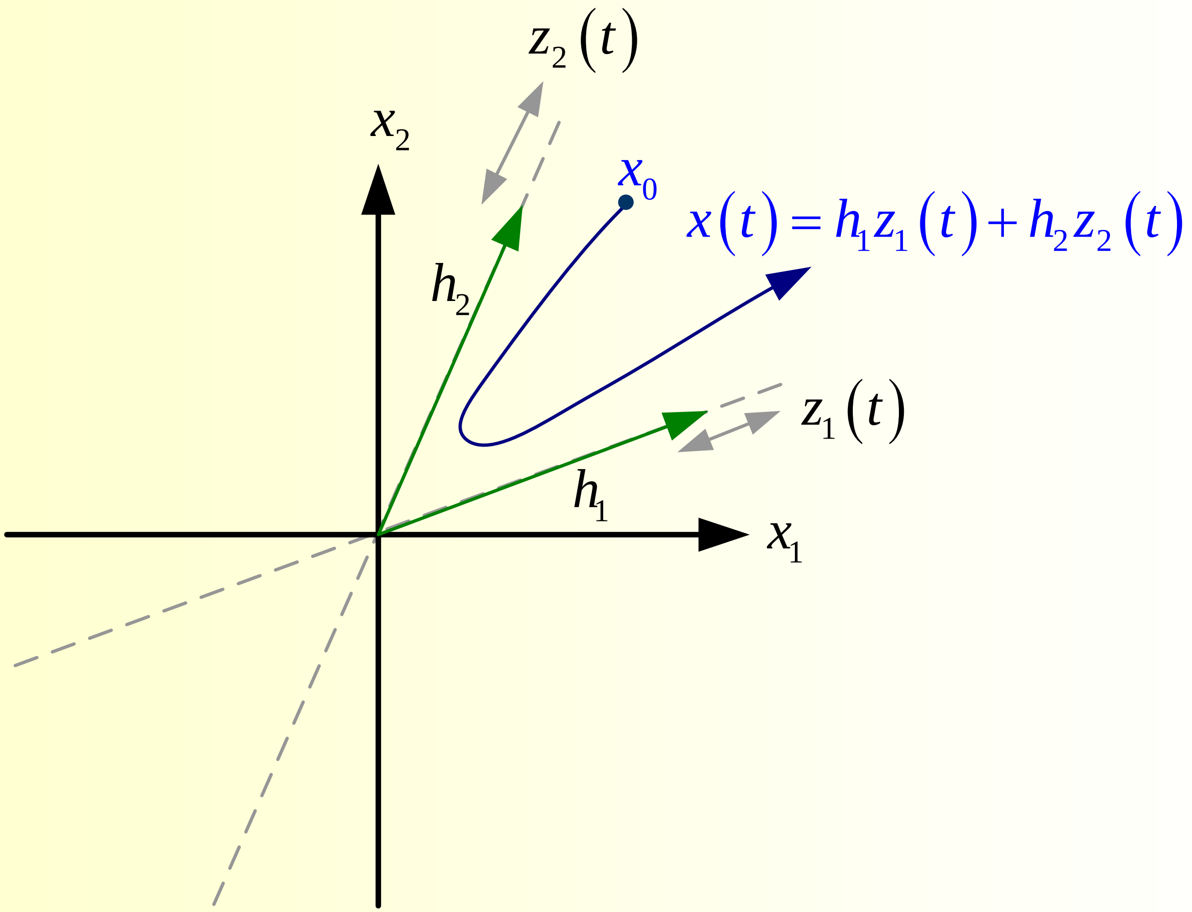
$$\dot{z}_i = \lambda_i z_i + \bar{b}_i u,\ i = 1,\ldots,n$$

Suppose we solve these equations to obtain $z_i(t),\ i = 1,\ldots,n,$ the we can obtain the solution in the original coordinates via

$$x(t) = Tz(t) = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = h_1 z_1(t) + h_2 z_2(t) + \cdots + h_n z_n(t)$$

Notice that each term $h_i z_i(t)$ describes a motion that takes place in the line defined by h_i.

Model Coordinates, 2



Modal Coordinates, 3

Suppose the system is unforce, $u(t) = 0$

The initial conditions for the modal coordinates are $z_0 = T^{-1}x_0$

The solution is

$$z_i(t) = e^{\lambda_i t} z_{0,i},\ i = 1, \ldots, n$$

$$x(t) = h_1 e^{\lambda_1 t} z_{0,1} + \cdots + h_n e^{\lambda_n t} z_{0,n}$$

The modes are the vector time functions $h_i e^{\lambda_i t}$, h_i is the mode shape.

The solution is a linear combination of the modes.

If λ_i is real, so is h_i and the modal response is a simple exponential.

If there are n linearly independent eigenvectors, then the set of solutions

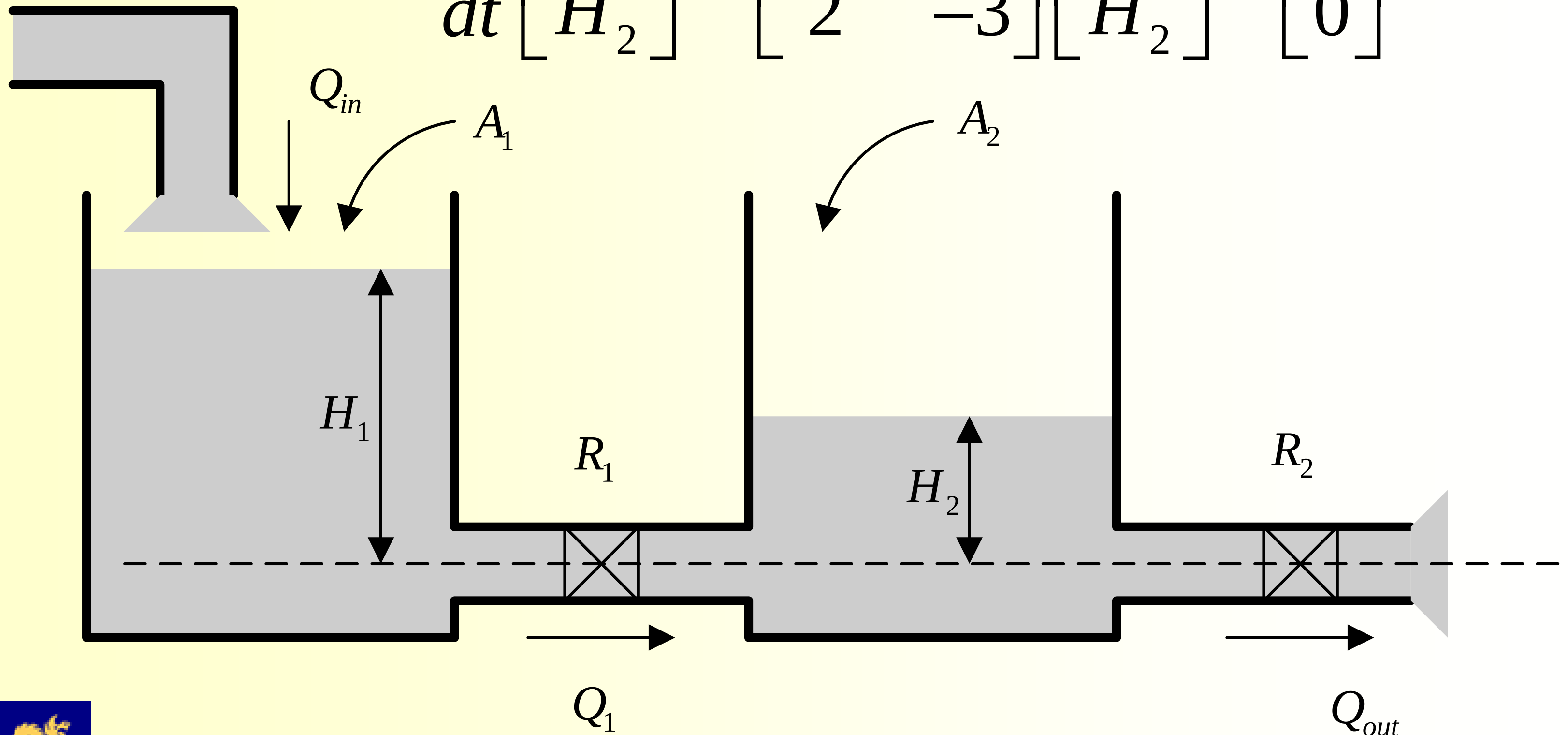
$$\left\{ h_1 e^{\lambda_1 t}, \ldots, h_n e^{\lambda_n t} \right\}$$

is called a fundamental set of solutions. Any solution to the homogeneous equation $\dot{x} = Ax$ is a linear combination of the fundamental solutions.

Example

$$A_1 = 1,\ A_2 = 1/2,\ R_1 = R_2 = 1$$

$$\frac{d}{dt}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Q_{in}$$



Example Cont'd

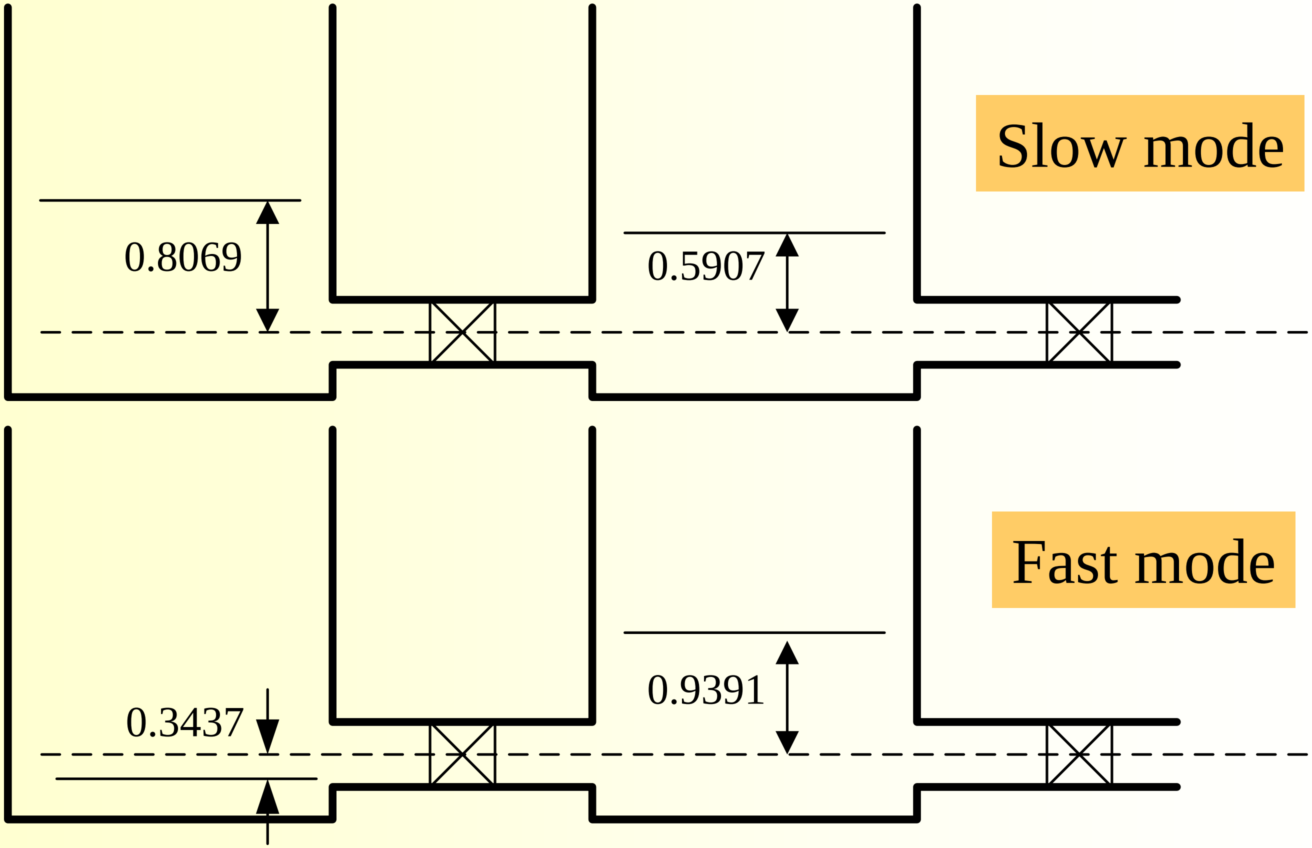
```
>> A=[-1 1;2 -3];
>> [E,V]=eig(A)
E =
     0.8069   -0.3437
     0.5907    0.9391
V =

    -0.2679         0
          0   -3.7321
```

$$\text{slow mode: } \lambda_1 = -0.2679, h_1 = \begin{bmatrix} 0.8069 \\ 0.5907 \end{bmatrix}$$

$$\text{fast mode: } \lambda_2 = -3.7321, h_2 = \begin{bmatrix} -0.3437 \\ 0.9391 \end{bmatrix}$$

Example, Cont'd



Complex Modes, 1

If the an eigenvalue λ_i is complex, the so is the eigenvector h_i.
To make things easier to interpret, we can construct real ones.
Suppose $\lambda_1 = \sigma + j\omega$ is a complex eigenvalue, and $\lambda_2 = \lambda_1^*$. Then the corresponding eigenvectors are $h_1, h_2 = h_1^*$. Correspondly, the two fundamental solutions are $h_1 e^{\lambda_1 t}, h_1^* e^{\lambda_1^* t}$. They are complex.
We will replace them with real ones.

Complex Modes, 2

Define

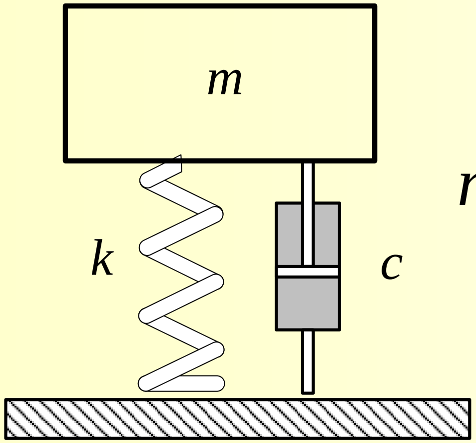
$$x_1(t) = \frac{1}{2}\left(h_1 e^{\lambda_1 t} + h_1^* e^{\lambda_1^* t}\right)$$

$$= \frac{1}{2}\begin{pmatrix} \left(h_{1,R} + jh_{1,I}\right)e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right) \\ +\left(h_{1,R} - jh_{1,I}\right)e^{\sigma t}\left(\cos\omega t - j\sin\omega t\right) \end{pmatrix}$$

$$= h_{1,R}e^{\sigma t}\cos\omega t - h_{1,I}e^{\sigma t}\sin\omega t$$

$$x_2(t) = \frac{1}{2}\left(-h_1 e^{\lambda_1 t} + h_1^* e^{\lambda_1^* t}\right) = h_{1,R}e^{\sigma t}\sin\omega t + h_{1,I}e^{\sigma t}\cos\omega t$$

Example



$$\dot{x} = v$$

$$m\dot{v} = -cv - kx$$

$$m = 1, k = 4, c = 1/2$$

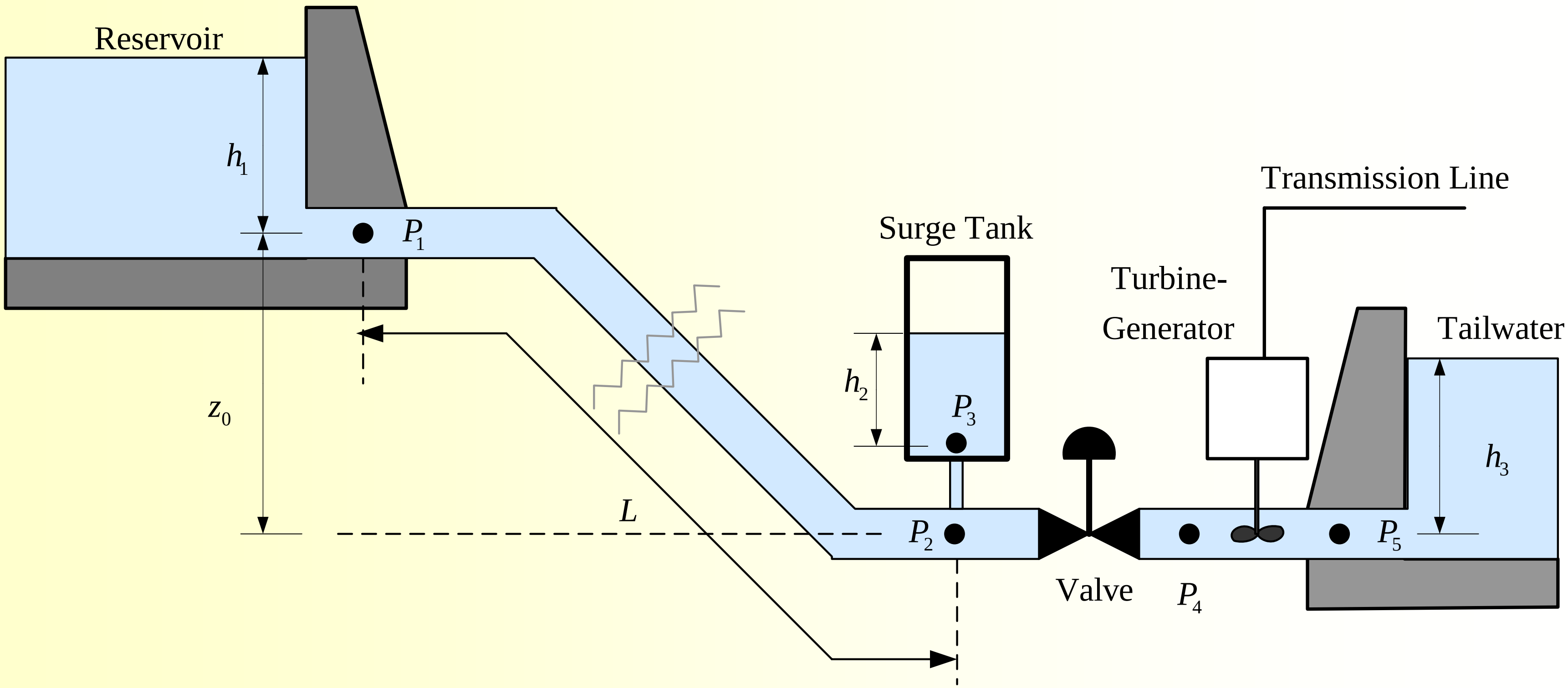
$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -0.5 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$h_{1,R} = \begin{bmatrix} 0.0559 \\ -0.8944 \end{bmatrix},$$

$$h_{1,I} = \begin{bmatrix} 0.4437 \\ 0 \end{bmatrix}$$

```
>> A=[0 1;-4 -0.5];
>> [E,V]=eig(A)
E =
   0.0559 + 0.4437i   0.0559 - 0.4437i
  -0.8944            -0.8944
V =
  -0.2500 + 1.9843i        0
       0             -0.2500 - 1.9843i
```

System



Reservoir
h_1
P_1
z_0
L
Surge Tank
h_2
P_3
P_2
Valve
P_4
Transmission Line
Turbine-
Generator
Tailwater
h_3
P_5

Differential Equations

$$\dot{h}_2 = \frac{Q_s}{4},$$

$$\dot{Q} = -P_2 / 2 \times 10^5$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{10.2 P_4 + 9.8 \times 10^5 Q_t - 10^7 \theta - 10^7 \omega}{9,425,000}$$

Remaining Elements

Splitter $Q = Q_s + Q_t$ (mass balance)

Orifice $Q_s = C_s \left(P_2 - k_s h_2 \right)$ (flow accross orifice $P_3 = k_s h_2$)

Valve $Q_t = \ell\left(t\right) Q_{t0} + \frac{1}{2} C_t \left(P_2 - P_4 \right)$ (throttle valve charateristic)

Turbine $P_4 - P_5 = C_q Q_t + C_\omega \omega$ (hydro machine characteristic)

Find: Q_s, Q_t, P_2, P_4

Recall the variables $Q, h_2, \omega, \ell\left(t\right)$ are treated as known.

Sol’ns to Algebraic Equations

$$P_2 = 9.09256h_2 - 0.0018512\ell + 0.0909256Q - 0.00907441\omega$$
$$P_4 = 9.07441h_2 + 0.0185118\omega + 0.0907441Q - 0.0092559\omega$$
$$Q_s = -9.07441h_2 - 0.0185118\ell + 0.909256Q - 0.0907441\omega$$
$$Q_t = 9.07441h_2 + 0.0185118\ell + 0.0907441Q + 0.0907441\omega$$

State Equations

$$\frac{d}{dt}\begin{bmatrix} h_2 \\ Q \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -2.2686 & 0.227314 & 0 & -0.022686 \\ -0.454628\times10^{-4} & -0.104546\times10^{-4} & 0 & 4.53721\times10^{-8} \\ 0 & 0 & 0 & 1 \\ 0.943556 & 9.43556\times10^{-3} & -1.06101 & -1.05157 \end{bmatrix}\begin{bmatrix} h_2 \\ Q \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} -0.00462795 \\ 9.2559^{-9} \\ 0 \\ 0.00192485 \end{bmatrix}\ell$$

Modes

-2.25582	$-0.532+i0.8853$	$-0.532-i0.8853$	-0.150×10^{-4}	
0.851	–0.007	0.004	0.099	h_2
0.0	0.0	0.0	0.990	Q
0.213	–0.358	–0.596	0.097	θ
–0.480	0.718	0.0	0.0	ω

The Matrix Exponential Revisited

Recall the matrix exponential:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

Now, suppose T is the transformation matrix that diagonalizes the matrix A,

$$T^{-1}AT = \Lambda \triangleq \operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)$$

Construct $T^{-1}e^{At}T$

$$\begin{aligned} T^{-1}e^{At}T &= T^{-1}IT + T^{-1}ATt + \frac{1}{2}T^{-1}A^2Tt^2 + \cdots + \frac{1}{k!}T^{-1}A^kTt^k + \cdots \\ &= I + \Lambda t + \frac{1}{2}T^{-1}ATT^{-1}ATt^2 + \cdots + \frac{1}{k!}\left(T^{-1}AT\right)^k t^k + \cdots \\ &= I + \Lambda t + \frac{1}{2}\Lambda^2t^2 + \cdots + \frac{1}{k!}\Lambda^kt^k + \cdots \\ &= \operatorname{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right) \end{aligned}$$

$$\boxed{e^{At} = T\operatorname{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right)T^{-1}}$$

Summary

- Similarity transformations
- Diagonalization using Eigenvectors
- Modes & Modal coordinates
- Interpreting behavior in terms of modes
- Converting complex modes into real ones

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