MEM 255 Introduction to Control Systems: *Stability & Parameter Dependant Dynamics*

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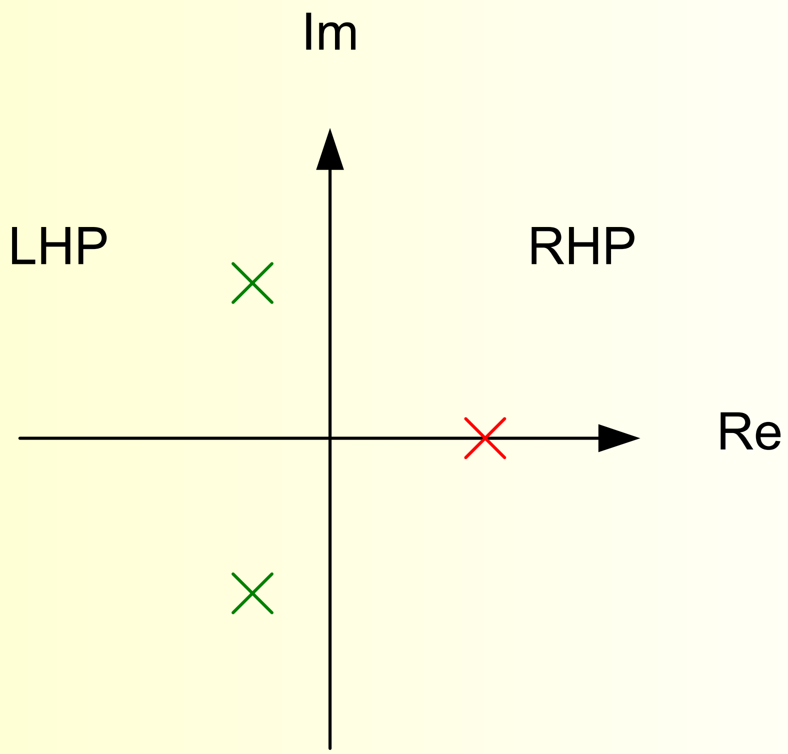
Outline

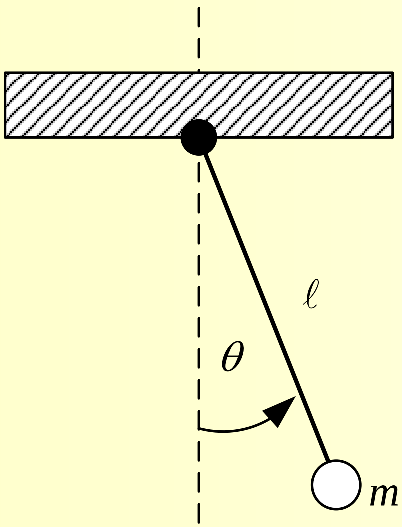
- Stability
  - Stability of linear time-invariant systems
  - Linear stability theory is useful for nonlinear analysis
- Examples
- Routh Hurwitz
  - This is a toll for investigating how stability is affected by change of a single parameter.
- Examples

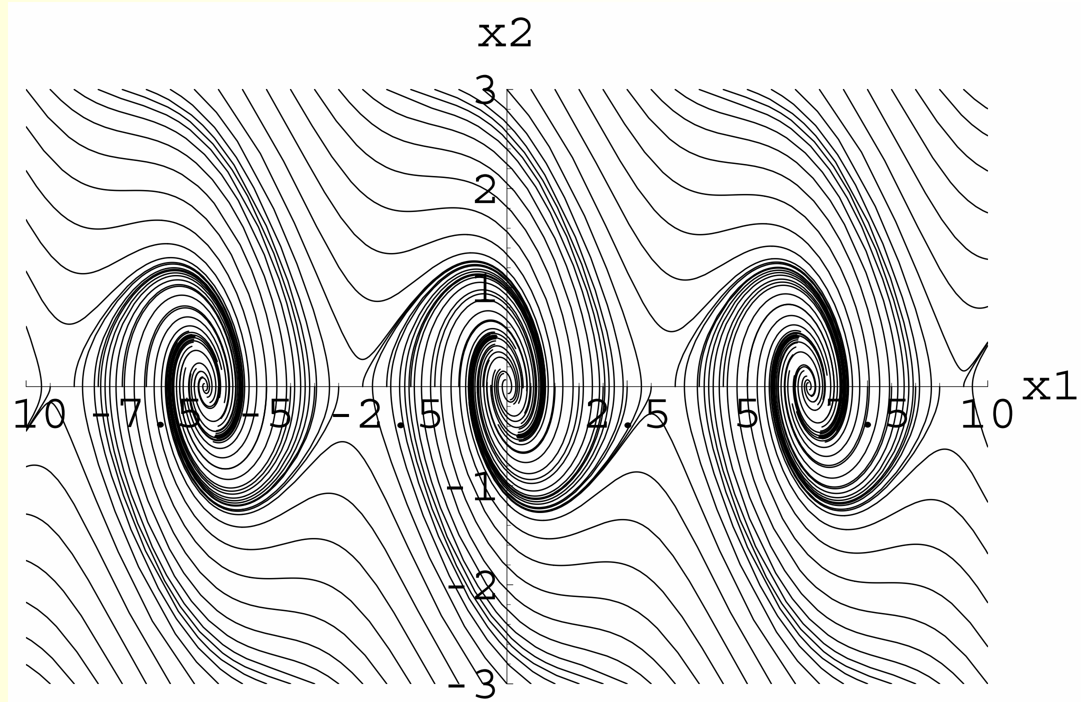
Stability

A linear time-invariant system is stable if all of its poles (eigenvalues) have negative real parts. Otherwise it is unstable.



Pendulum





$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta - c\dot{\theta} + T(t)$$

$$x_1 \triangleq \theta,\ x_2 \triangleq \dot{\theta} \Rightarrow$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(g/\ell)\sin\theta - \left(c/m\ell^2\right)\dot{\theta} + \left(1/m\ell^2\right)T(t)\end{aligned}$$

Linear stability analysis is useful in nonlinear system studies. Linear approximations at equilibrium points are used to establish their stability properties.

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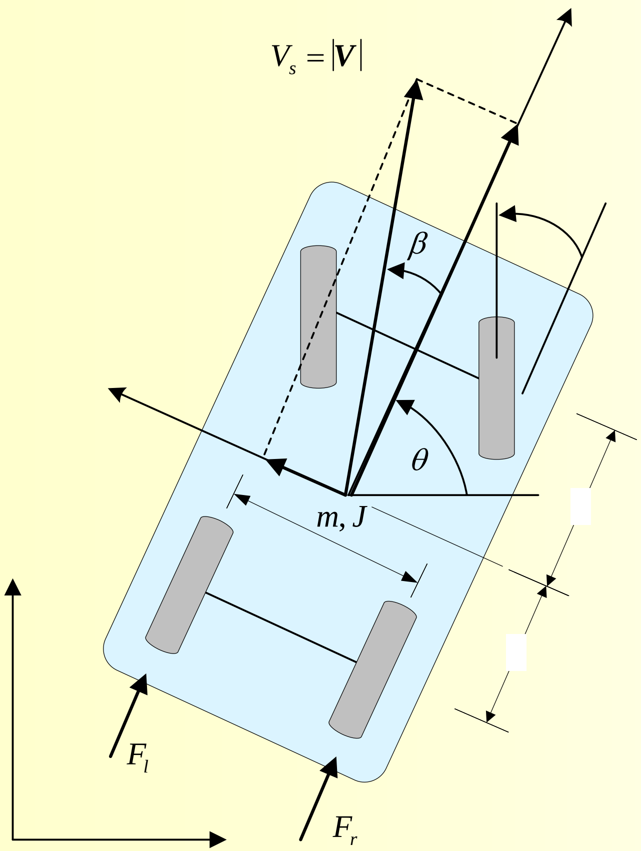
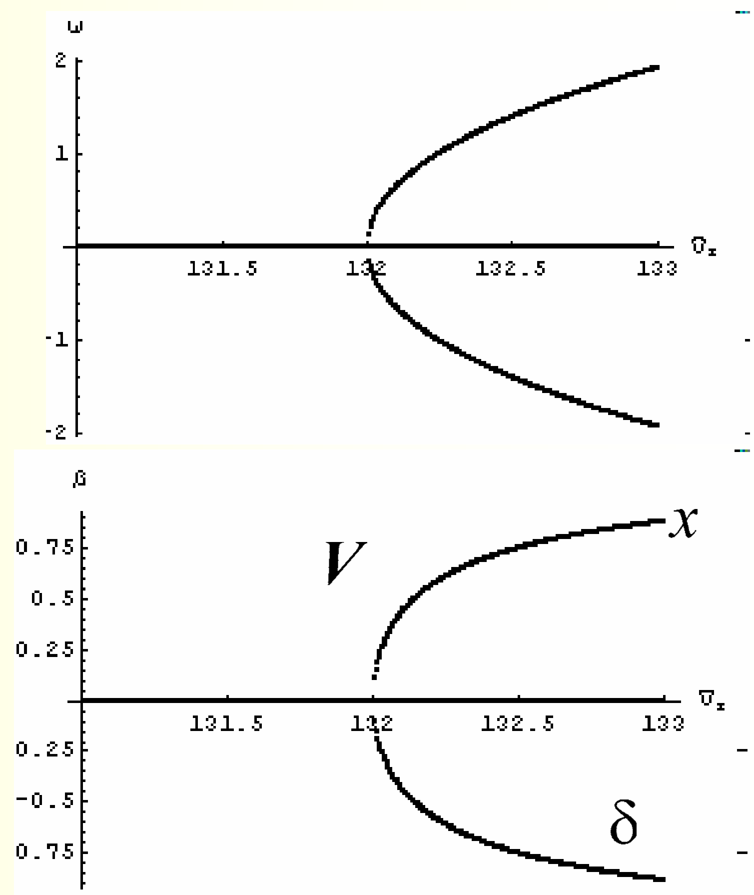
Linear analysis can only go so far. Stability changes that take place when parameters vary are often associated with changes in the number of equilibrium points or the appearance of other kinds of invariant sets.





Routh Hurwitz Analysis

$$\phi(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

$$
\begin{array}{c|cccc}
s^4 & a_4 = 1 & a_2 & a_0 & 0 \\
s^3 & a_3 & a_1 & 0 & 0 \\
\hline
s^2 & \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & \dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 & \dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0 & \\
s^1 & \dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0 & & \\
s^0 & \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 & & &
\end{array}
$$

The number of roots in the right half plane equals the number of sign changes in the first column.

Simple Example

$$\phi(s) = s^3 + 2\rho s^2 + s, \quad \rho \neq 0$$

$$\begin{array}{llll} s^3 & 1 & 1 & 0 \\ s^2 & 2\rho & 0 & 0 \\ s^1 & 1 & 0 & \\ s^0 & 0 & & \end{array} \Rightarrow \begin{array}{l} \text{for } \rho > 0, \text{ there are 0 roots in RHP} \\ \text{for } \rho < 0, \text{ there are 2 roots in RHP} \end{array}$$

Example: Robotic Arm

$$G_{cl} = \frac{20K(s+0.1)}{s^5 + 15s^4 + 74.25s^3 + 121s^2 + 20Ks + 2K}$$

Routh Table

s^5	1	74	$20K$
s^4	15	121	$2K$
s^3	65.9	$19.86K$	
s^2	$121-4.52K$	$2K$	
s^1	$\frac{2271K - 89.76K^2}{121-4.52K}$		
s^0	$2K$		

Governing Inequality

$$121-4.52K > 0,\ 2271K - 89.76K^2 > 0,\ K > 0$$

$$K < \frac{121}{4.52} = 26.769$$

$$\boxed{K < \frac{2271}{89.76} = 25.308}$$

Example: Boeing 747

Level flight at 40,000 ft, 744 ft/sec (Mach 0.8)

$$\frac{d}{dt}\begin{bmatrix}\beta \\ r \\ p \\ \phi\end{bmatrix} = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix}\begin{bmatrix}\beta \\ r \\ p \\ \phi\end{bmatrix} + \begin{bmatrix} 0.00729 \\ -0.475 \\ 0.153 \\ 0 \end{bmatrix}\delta_r$$

Modes

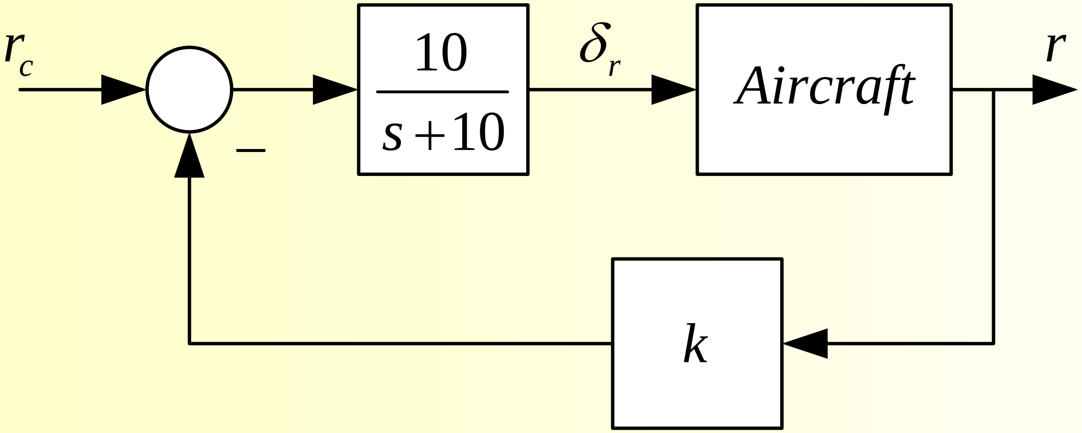
```
>> A=[-0.0558 -0.9968 0.0802 0.0415
      0.598 -0.115 -0.0318 0
      -3.05 0.388 -0.4650 0
       0 0.0805 1 0];
>> [E,V]=eigs(A)
E =
   0.1994 + 0.1063i   0.1994 - 0.1063i  -0.0172             0.0067
  -0.0780 + 0.1333i  -0.0780 - 0.1333i  -0.0118             0.0404
  -0.0165 - 0.6668i  -0.0165 + 0.6668i  -0.4895            -0.0105
   0.6930             0.6930             0.8717             0.9991
V =
  -0.0329 - 0.9467i        0                  0                  0
        0            -0.0329 + 0.9467i        0                  0
        0                  0            -0.5627                  0
        0                  0                  0            -0.0073
```

Dutch roll

Roll

Spiral

Yaw Stabilizer



$$G_1(s) = \frac{10}{s+10} G_{aircraft}(s) = \frac{n(s)}{d(s)}$$

$$G_{cl}(s) = \frac{G_1(s)}{1+kG_1(s)} = \frac{n(s)}{d(s)+kn(s)}$$

Transfer Function

```
>> B=[0.00729;-0.475;0.153;0];
>> C=[0 1 0 0];
>> S=ss(A,B,C,0)
>> G=zpk(S)
Zero/pole/gain:
   -0.475 (s+0.4981) (s^2  + 0.02379s + 0.2381)
--------------------------------------------------
(s+0.5627) (s+0.007278) (s^2  + 0.06587s + 0.8972)
>> Gr=10/(s+10);
>> tf(series(Gr,G))
Transfer function:
        -4.75 s^3 - 2.479 s^2 - 1.187 s - 0.5633
--------------------------------------------------------
s^5 + 10.64 s^4 + 7.297 s^3 + 9.9 s^2 + 5.12 s + 0.03674
```

Characteristic Polynomial

```
>> syms s k
>> d1=s^5 + 10.64*s^4 + 7.297*s^3 + 9.9*s^2 + 5.12*s + 0.03674;
>> n1=-4.75*s^3 - 2.479+s^2 - 1.187*s - 0.5633
>> digits(5)
>> vpa(collect(d1+k*n1))
ans =
s^5+10.640*s^4+(-4.7500*k+7.2970)*s^3+(k+9.9000)*s^2+(5.1200-
1.1870*k)*s+.36740e-1-3.0423*k
```

Routh Table

$$
\begin{array}{c|cccc}
s^5 & 1 & 7.30-4.75k & 5.12-1.19k & 0 \\
s^4 & 10.64 & 9.9+k & 0.0367-3.04k & 0 \\
\hdashline
s^3 & b_1 & b_2 & 0 & 0 \\
s^2 & c_1 & c_2 & 0 & \\
s^1 & d_1 & 0 & 0 & \\
s^0 & e_1 & & &
\end{array}
$$

Routh Table,2

```
)1 = Simplify[-((9.9 + k) - 10.64 (7.297 - 4.75 k)) / 10.64]

;.36655 - 4.84398 k

)2 =
 Simplify[-((.03674 - 3.0423 k) - 10.64 (5.12 - 1.19 k)) /
   10.64]

;.11655 - 0.90407 k

:1 = Simplify[-(10.64 b2 - (9.9 + k) b1) / b1]

  4.84398 (-0.258527 + k) (6.85838 + k)
- -------------------------------------
          6.36655 - 4.84398 k

:2 = Simplify[-(10.64 0 - (.03674 - 3.0423 k) b1) / b1]

).03674 - 3.0423 k

l1 = Simplify[-(b1 c2 - b2 c1) / c1]

  15.6409 (-2.58355 + k) (-0.396798 + k) (0.54662 + k)
- ----------------------------------------------------
            (-0.258527 + k) (6.85838 + k)

:1 = Simplify[-(-c2 d1) / d1]

).03674 - 3.0423 k
```

Inequalities

$$b_1 \rightarrow 6.367 - 4.844k > 0$$

$$c_1 \rightarrow -(-0.259 + k)(6.858 + k) > 0$$

$$d_1 \rightarrow (-0.2584 + k)(-0.3968 + k)(0.5466 + k) > 0$$

$$e_1 \rightarrow (0.3674 - 3.0423k) > 0$$

$$-0.547 < k < 0.0121$$

Summary

- A linear time-invariant systems is stable if all of its poles are in the LHP. Otherwise it is unstable.
- Linear theory can be used to determine the stability of equilibrium points of a nonlinear system. But there are other aspects of nonlinear system that it does not reveal.
- The Routh-Hurwitz method is a tool that can be used to determine the values of a parameter for which a linear system is stable.