MEM 255 Introduction to Control Systems

Review: Complex Numbers

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Outline

- Definition, & Representation
- Basic Properties
- Euler's Formula
- Algebraic Operations
- Functions of a Complex Variable
- MATLAB Functions

Complex Numbers

- A **complex number** is defined as

$$s = x + jy \quad (\text{or } x + iy)$$

where $j = \sqrt{-1}$ $\left(\text{or } i = \sqrt{-1}\right)$ is the **imaginary operator** and x, y are real numbers.

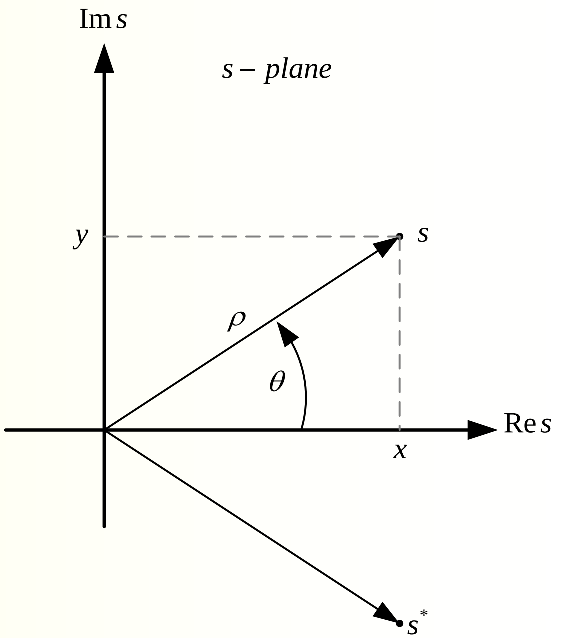
- x is the real part of s : $x = \text{Re}\, s$

y is the imaginary part of s : $y = \text{Im}\, s$

- ρ, θ defined by $x = \rho\cos\theta$, $y = \rho\sin\theta$ are called the magnitude and angle of s, respectively. Note

$$\rho = |s| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)$$

- $s^* = x - jy$ is called the **complex conjugate** of s



Basic Properties of Complex Numbers

$$\left(s^*\right)^* = s$$

$$\left(s_1 \pm s_2\right)^* = s_1^* \pm s_2^*$$

$$\left(\frac{s_1}{s_2}\right)^* = \frac{s_1^*}{s_2^*}$$

$$\left(s_1 s_2\right)^* = s_1^* s_2^*$$

$$\operatorname{Re} s = \frac{s + s^*}{2}, \quad \operatorname{Im} s = \frac{s - s^*}{2}$$

$$s s^* = \left|s\right|^2$$

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Euler's Formula

It is easy to prove the fundamental relationship:

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

To see this, set

$$A(\theta) = \cos\theta + j\sin\theta$$

We can derive and solve a differential equation for A

$$\frac{dA}{d\theta} = -\sin\theta + j\cos\theta = j(\cos\theta + j\sin\theta) = jA$$

$$\frac{dA}{A} = jd\theta \Rightarrow \ln A = j\theta + c$$ where c is a constant of integration

$$\Rightarrow A = e^{j\theta+c} = e^{j\theta}e^{c}$$

To determine c, evaluate $e^{j\theta}e^{c} = \cos\theta + j\sin\theta$ at $\theta = 1$;

$$\Rightarrow e^{c} = 1 \Rightarrow c = 0$$

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Polar Representation

Consider the complex number *s*:

$$s = x + jy = \rho\cos\theta + j\rho\sin\theta = \rho e^{j\theta}$$

rectangular form: $s = x + jy$

polar form: $s = \rho e^{j\theta}$

Many computations including multiplication and division are much easier to do in polar form.

Algebraic Operations

Consider two complex numbers

$$s_1 = x_1 + jy_1, \quad s_2 = x_2 + jy_2$$

addition: $s_1 + s_2 = \left(x_1 + x_2\right) + j\left(y_1 + y_2\right)$

multiplication: $s_1 s_2 = \left(x_1 x_2 - y_1 y_2\right) + j\left(x_1 y_2 + x_2 y_1\right)$

$$\Rightarrow s_1 s_1^* = x_1^2 + y_1^2$$

division: $\dfrac{s_1}{s_2} = \dfrac{s_1}{s_2}\dfrac{s_2^*}{s_2^*} = \dfrac{\left(x_1 x_2 + y_1 y_2\right) + j\left(-x_1 y_2 + x_2 y_1\right)}{x_2^2 + y_2^2}$

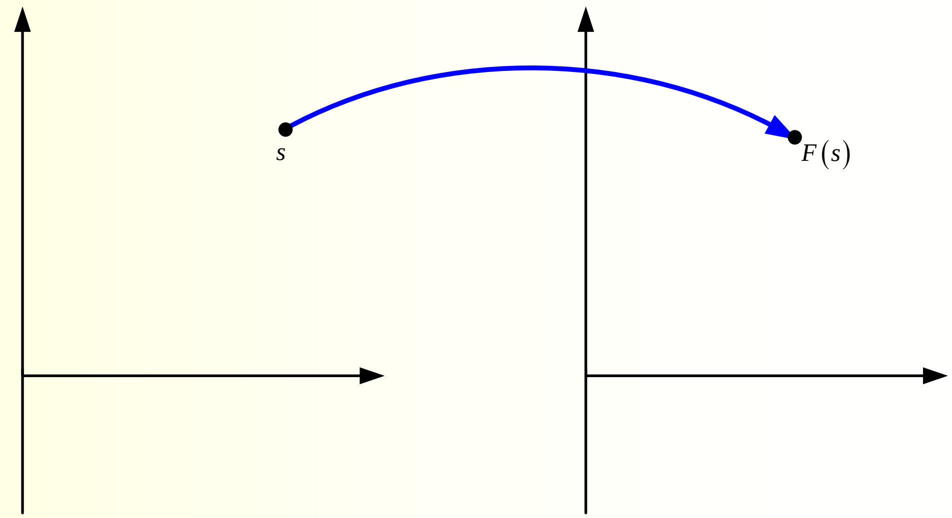
Functions of a Complex variable

A function F of a complex variable s can be thought of as a mapping from one complex plane (s-plane) to another (F-plane).

The function F is **analytic** at a point s_0 the s-plane if its derivative dF/ds exists at all points in a neighborhood of s_0.

The function F is called a **rational function** if it is the ratio of two polynomials in s, e.g.

$$F(s) = \frac{(s+1)(s-2)}{(s+2)(s^2+9)}$$



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Singularities

If a function $F(s)$ is not analytic at a point s_0, then s_0 is a **singular point**.

If a function $F(s)$ is not analytic at a point s_0, but is analytic at every other point in a neighborhood of s_0 then s_0 is an **isolated** singular point.

Example: $F(s) = \frac{1}{s}$ has an isolated singular point at $s = 0$

If $F(s)$ is a rational function, then the only singularities are the roots of the denominator (called poles).

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MATLAB Functions

Function	Returns
imag	imaginary part of a complex number
real	real part of a complex number
abs	magnitude of a complex number
angle	angle of a complex number
conj	conjugate of a complex number

Advanced Topics ~ Defer

- Cauchy's Residue Theorem
- Principle of the Argument

Summary

- Complex numbers
  - Rectangular and polar forms
  - Basic algebra
- Euler's formula
- Functions of a complex variable
  - Rational functions
  - Singularities & poles
- MATLAB tools

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