MEM 255 Introduction to Control Systems

Laplace Transforms and Transfer Functions

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Outline

- **Laplace Transforms**
  - **Definition**
  - **Short table of transform pairs**
  - **Inversion via partial fraction expansion**
  - **Using MATLAB symbolics for Laplace transforms**
- **Linear Systems**
  - **State space and transfer function models**
  - **Solving linear odes with the Laplace transform**
  - **Computing transfer functions with MATLAB**

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Laplace Transform Definition

$$f(t) = \begin{cases} 0 & t < 0 \\ \text{piecewise continuous} & t \geq 0 \end{cases}$$

$$F(s) = L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt$$

$$f(t) = L^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} e^{st} F(s)\,ds$$

Example:

$$f(t) = e^{-\lambda t}u(t) \Rightarrow F(s) = \int_0^\infty e^{-(\lambda+s)t}\,dt \quad \left(\mathrm{Re}(\lambda+s) > 0\right)$$

$$= \left.\frac{e^{-(\lambda+s)t}}{-(\lambda+s)}\right|_0^\infty = 0 - \frac{-1}{(\lambda+s)} = \frac{1}{s+\lambda}$$

Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-\lambda t}u(t)$	$\frac{1}{s+\lambda}$
$\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\,u(t)$	$\frac{s}{s^2+\omega^2}$

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Basic Theorems

Linearity	$L[\alpha_1 f_1(t) + \alpha_2 f_2(t)] = \alpha_1 F_1(s) + \alpha_2 F_2(s)$
Time Shift	$L[f(t-T)] = e^{-sT} F(s)$
Frequency Shift	$L[e^{-at} f(t)] = F(s+a)$
Derivative	$L[\dot{f}(t)] = sF(s) - f(0)$
Final Value	$f(\infty) = \lim_{s \to 0} sF(s)$
Initial Value	$f(0+) = \lim_{s \to \infty} sF(s)$

Partial Fraction Expansion, 1

$$Y(s) = k\frac{n(s)}{d(s)} = k\frac{\textit{monic poly of } \deg = m}{\textit{monic poly of } \deg = n > m}$$

$$n(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_0 = (s-\zeta_m)(s-\zeta_{m-1})\cdots(s-\zeta_1)$$

$$d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = (s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_1)$$

$$Y(s) = \frac{k\,n(s)}{(s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_1)}$$

Partial Fraction Expansion, 2

Case 1: distinct roots, $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

When the denominator has n distinct roots, the transfer function can be expanded in the form with a unique set if coefficients, $c_i, i = 1, \ldots, n$. These constants are called **residues**.

$$Y(s) = \frac{k\,n(s)}{(s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_1)} = \frac{c_1}{s-\lambda_1} + \cdots + \frac{c_n}{s-\lambda_n}$$

To determine c_i

$$\frac{(s-\lambda_i)k\,n(s)}{(s-\lambda_n)\cdots(s-\lambda_i)\cdots(s-\lambda_1)} = \frac{(s-\lambda_i)c_1}{s-\lambda_1} + \cdots + \frac{(s-\lambda_i)c_i}{(s-\lambda_i)} + \cdots + \frac{(s-\lambda_i)c_n}{s-\lambda_n}$$

$$\frac{k\,n(s)}{(s-\lambda_n)\cdots 1 \cdots(s-\lambda_1)} = \frac{(s-\lambda_i)c_1}{s-\lambda_1} + \cdots + c_i + \cdots + \frac{(s-\lambda_i)c_n}{s-\lambda_n}$$

Now set $s \to \lambda_i$, to obtain

$$c_i = \lim_{s \to \lambda_i} (s-\lambda_i)Y(s)$$

Partial Fraction Example

$$Y(s) = \frac{5}{s^2 + 3s + 2} = \frac{5}{(s+1)(s+2)}$$

$$Y(s) = \frac{c_1}{s+1} + \frac{c_2}{(s+2)}$$

$$c_1 = (s+1)Y(s)\Big|_{s \to -1} = \frac{5}{s+2}\Bigg|_{s \to -1} = 5$$

$$c_2 = (s+2)Y(s)\Big|_{s \to -1} = \frac{5}{s+1}\Bigg|_{s \to -2} = -5$$

$$Y(s) = \frac{5}{s+1} + \frac{-5}{(s+2)}$$

$$y(t) = L^{-1}[Y(s)] = \left(5e^{-t} - 5e^{-2t}\right)u(t) \quad \left(1\big/(s+\lambda) \Leftrightarrow e^{-\lambda t}u(t)\right)$$

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Symbolic Computing

MATLAB/
Maple

```
>> syms a w t s
>> f1=exp(-a*t)*sin(w*t);
>> F1=laplace(f1)
F1 =
w/((s+a)^2+w^2)
>> f10=ilaplace(F1)
f10 =
w/(-4*w^2)^(1/2)*(exp((-a+1/2*(-4*w^2)^(1/2))*t)
                  -exp((-a-1/2*(-4*w^2)^(1/2))*t))
```

```
f1 = Exp[-a t] Sin[ω t];

F1 = LaplaceTransform[f1, t, s]
```

$$\frac{\omega}{a^2 + 2 a s + s^2 + \omega^2}$$

```
InverseLaplaceTransform[F1, s, t]
```

$e^{-a t}$ Sin[t ω]

Mathematica

Computing the Partial Fraction Expansion

```
>> s=tf('s');
>> sys=10*s*(s+4)/((s+2)*(s^2+s+5))
Transfer function:
    10 s^2 + 40 s
----------------------
s^3 + 3 s^2 + 7 s + 10
>> [num,den]=tfdata(sys,'v');
>> [r,p,k]=residue(num,den)
r =
   7.8571 - 1.4748i
   7.8571 + 1.4748i
  -5.7143
p =
  -0.5000 + 2.1794i
  -0.5000 - 2.1794i
  -2.0000
k =
     []
```

$$\frac{10s^2+40s}{s^3+3s^2+7s+10} = \frac{7.8571-1.470i}{s+0.5-2.1794i}+\frac{7.8571+1.470i}{s+0.5+2.1794i}+\frac{-5.7143}{s+2}$$

Summary

- Laplace transforms
  - Transform pairs
  - Basic theorems
  - Inversion using partial fraction expansion
  - Symbolic computing tools