# MEM 255 Introduction to Control Systems

## *Review: Basics of Linear Algebra*

Harry G. Kwatny

Department of Mechanical Engineering & Mechanics

Drexel University

# Outline

- Vectors
- Matrices
- MATLAB
- Advanced Topics

# Vectors

A vector is a one-dimensional array of scalarelements (real or complex numbers)

column vector: $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, row vector: $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$

Vectors of equal dimension can be added (elementwise):

$$x = a + b \Leftrightarrow x_i = a_i + b_i, \quad i = 1, \ldots, n$$

Vectors can be multiplied by a scalar:

$$x = \alpha a \Leftrightarrow x_i = \alpha a_i$$

Vectors can be transposed:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

# Inner Product & Norm

The **inner product** of two $n$-dimensional vectors $x, y$ is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

for column vectors $\langle x, y \rangle = x^T y$

for row vectors $\langle x, y \rangle = x y^T$

The **Euclidean norm** or length of a vector $x$ is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Other norms or length measures are also just useful:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_\infty = \max_i |x_i|$$

# Linear Combinations of Vectors

Suppose $\alpha_i, i = 1, \ldots, p$ is a set of scalars and $x_i, i = 1, \ldots, p$ is a set of column or row vectors, then we define a new vector $y$ via the linear combination:

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p$$

for columns

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \alpha_1 \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{1,n} \end{bmatrix} + \cdots + \alpha_p \begin{bmatrix} x_{p,1} \\ \vdots \\ x_{p,n} \end{bmatrix}$$

A set of $p$ vectors $x_i, i = 1, \ldots, p$ is **linearly dependent** if there exists a nontrivial set of constants $\alpha_i, i = 1, \ldots, p$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p = 0$$

otherwise it is **linearly independent**.

A set of linearly independent $n$-dimensional vectors contains at most $n$ vectors.

# Matrices

- A *matrix* is a 2-dimensional (rectangular) array of elements:

$$m \text{ rows} \left\{ \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}}_{n \text{ columns}} \right.$$

Sometimes we write $A = \left[ a_{ij} \right]$

- The elements are called *scalars*, they are usually real or complex numbers.

- A matrix with one row, $m = 1$, is called a *row matrix* or *row vector*.

A matrix with one column, $n = 1$, is called a *column matrix* or *column vector*.

# Algebraic Operations

- Equality - two matrices (of the same size) $A, B$ are equal, written $A = B$, if their corresponding elements are equal, $a_{ij} = b_{ij}$, for $1 \le i \le m, 1 \le j \le n$
- Matrices of the same size can be added and subtracted. Matrix addition and subtraction are performed element-wise

$$A + B = C \Leftrightarrow a_{ij} + b_{ij} = c_{ij} \qquad A - B = C \Leftrightarrow a_{ij} - b_{ij} = c_{ij}$$

- Any matrix $A = \left[ a_{ij} \right]$ can be multiplied by a scalar $\alpha$

$$\alpha A = \left[ \alpha a_{ij} \right]$$

- An $m \times n$ matrix $A$ can be post multiplied by an $n \times q$ matrix $B$ to produce an $m \times q$ matrix $C$,

$$C = AB, \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

Drexel UNIVERSITY

# Multiplication

$$\begin{bmatrix} \times & \times \\ \times & c_{22} \\ \times & \times \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline \times & \times & \times & \times \end{bmatrix} \left[\begin{array}{c|c} \times & b_{12} \\ \times & b_{22} \\ \times & b_{23} \\ \times & b_{24} \end{array}\right]$$

$$\Updownarrow$$

$$c_{22} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \end{bmatrix} = \sum_{k=1}^{4} a_{2k} b_{k2}$$

Drexel UNIVERSITY

# Transpose

The *transpose* of an $m \times n$ matrix is the $n \times p$ matrix obtained by interchanging rows and columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & & a_{m2} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

A matrix is *symmetric* if $A^T = A$

The following rules obtain:

$$(AB)^T = B^T A^T$$

$$(A+B)^T = A^T + B^T$$

# Determinant

- The $ij$-th **minor** $M_{ij}$ of a square $n \times n$ matrix $A$ is the $(n-1)\times(n-1)$ submatrix of $A$ obtained by eliminating row $i$ and column $j$.
- The **determinant** of a square matrix is defined recursively.

The determinant of a $1\times1$ matrix is $\det[a_{11}] = a_{11}$.

The determinant of a $n \times n$ matrix is defined by the expansion

$$\det A = \sum_{j=1}^{n} a_{ij}\gamma_{ij} \quad \text{for any } i = 1, 2, \ldots, n$$

where $\gamma_{ij}$ is the cofactor $\gamma_{ij} = (-1)^{i+j} \det M_{ij}$

*Note*: 'for any $i$' means expand along any row. The same result is obtained by expanding along any column.

# Properties of Determinants

- multiply any single row or column of $A$ by scalar $\alpha$ to get $\bar{A}$

  $\det \bar{A} = \alpha \det A$
- interchange any two rows or columns of $A$ to get $\bar{A}$

  $\det \bar{A} = -\det A$
- add multiple of any row or column to another row or column to get $\bar{A}$

  $\det \bar{A} = \det A$
- $\det A^T = \det A, \det AB = \det A \det B$
- for $A, C$ square

  $$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \det D$$
- for $A$ nonsingular

  $$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det \left[ D - CA^{-1}B \right]$$

# Matrix Inverse

An **identity matrix** of size $n$ is the square matrix with $n$ rows and columns:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

The **adjugate** of a square matrix $A$ is defined as the transpose of the matrix of cofactors:

$$\operatorname{adj} A = \left[\gamma_{ij}\right]^T$$

It can be shown that $A \operatorname{adj} A = (\det A) I$

If $\det A \neq 0$, we have $A \dfrac{\operatorname{adj} A}{\det A} = I$

A square matrix $A$ with $\det A \neq 0$ is called **nonsingular**, for a nonsingular matrix we can define the **inverse**

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} \Rightarrow AA^{-1} = I,\ A^{-1}A = I$$

# Rank

Consider an $m \times n$ matrix $A$.

- The number of linearly independent rows of $A$ equals the number of its linearly independent columns.
- The **rank** of $A$ is the number of its linearly independent rows or columns.
- $\boxed{\text{rank}\, A \leq \max(m,n)}$
- If $A$ is a square matrix of size $n$, $\boxed{\text{rank}\, A = n \Leftrightarrow \det A \neq 0}$

# MATLAB Basic Operations

+ Addition A and B must have the same size, unless one is a scalar. A scalar can be added to a matrix of any size.

- Subtraction A and B must have the same size, unless one is a scalar. A scalar can be subtracted from a matrix of any size.

* Matrix multiplication. For nonscalar A and B, the number of columns of A must equal the number of rows of B. A scalar can multiply a matrix of any size.

/ Slash or matrix right division. B/A is roughly the same as B*inv(A). More precisely, B/A = (A'\B')'.

\ Backslash or matrix left division. If A is a square matrix, A\B is roughly the same as inv(A)*B, except it is computed in a different way.

Drexel UNIVERSITY

# MATLAB Basic Operations

^ Matrix power. X^p is X to the power p, if p is a scalar. If p is an integer, the power is computed by repeated squaring. If the integer is negative, X is inverted first.

' Matrix transpose. A' is the linear algebraic transpose of A. For complex matrices, this is the complex conjugate transpose.

.' Array transpose. A.' is the array transpose of A. For complex matrices, this does not involve conjugation.

Note: The slash\backslash operations are the better than inv to solve linear equations.

# MATLAB Basic Functions

| | |
|---|---|
| norm | matrix or vector norm |
| rank | matrix rank |
| det | determinant |
| trace | sum of the diagonal elements |
| inv | matrix inverse |

Drexel UNIVERSITY

# Applications of Matrices

Matrices are important in many applications. One of the most important is the solution of sets of simultaneous linear equations:

$$\left.\begin{matrix} a_{11}x_1 + a_{22}x_2 + \cdots a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_2 \end{matrix}\right. \Rightarrow Ax = b, \text{ if } \det A \neq 0 \Rightarrow x = A^{-1}b$$

Another application is in the solution of sets of simultaneous linear ordinary differential equations, for example, equations like

$$\left.\begin{matrix} m\ddot{y} + c\dot{y} + ky = f(t) \\ \rho\dot{v} + \alpha y = g(t) \\ L\ddot{q} + C\dot{v} + B\dot{y} = h(t) \end{matrix}\right. \quad \text{can be put in the form } \dot{x} = Ax + b(t)$$

where $A$ is a properly defined square matrix, and $x, b$ are properly defined column vectors.

# Similarity Transformations

Sometimes it is useful to solve the equations in a coordinate system that is different from the original problem formulation. Any square nonsingular matrix $T$ can be considered a **transformation** matrix. Linear coordinate transformations of column vectors are accomplished via transformations

$$x = T\bar{x}, \quad \bar{x} = T^{-1}x$$

For example, under such a transformation

$$\dot{x} = Ax + b(t) \Rightarrow \dot{\bar{x}} = T^{-1}AT\bar{x} + T^{-1}b(t)$$

Matrices transform under a change of coordinates according to

$$\bar{A} = T^{-1}AT$$

This is called a **similarity transformation**.

# Special Matrices

Similarity transformations are used to transform matrices into a variety of special forms (when possible). Among these are:

diagonal: $\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$, upper triangular: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$

lower companion: $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$

# Advanced Topics ~ Defer

- Eigenvalues/Eigenvectors
- Functions of Matrices
- Cayley-Hamilton Theorem
- Singular Values

Drexel UNIVERSITY

# Summary

- Vectors
  - Basic definitions & operations
  - linear dependent\independent sets
- Matrices
  - Definitions
  - Algebraic operations
  - Determinants, Rank, Inverse
  - Similarity transformations & special matric forms
  - MATLAB functions