MEM 255 Introduction to Control Systems: *Modeling & analyzing systems*

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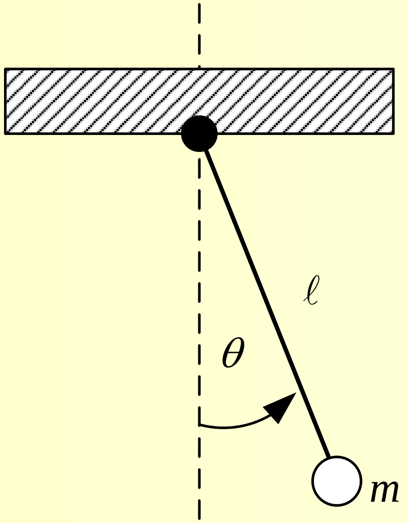
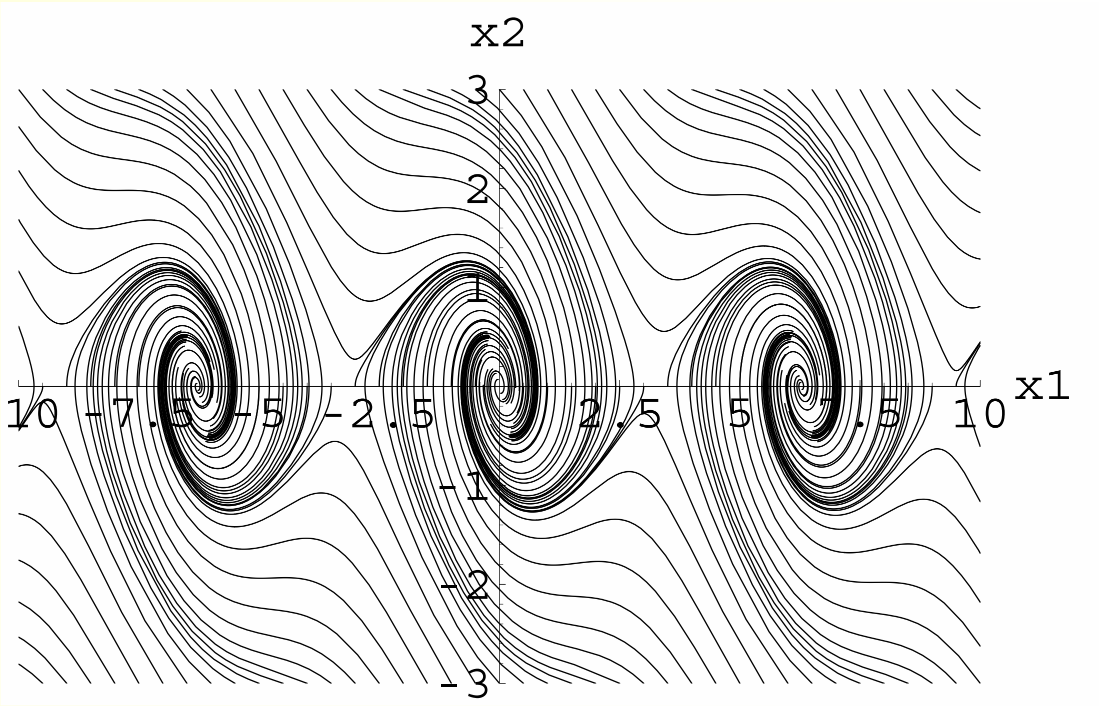
Outline

- **The Pendulum**
- **Micro-machined capacitive accelerometer**
  - **Differential equation model**
  - **Static (equilibrium) behavior**
  - **Dynamic behavior**
- **Linearizing Nonlinear Models**
- **Transfer Functions**
- **Using MATLAB numerics for solving odes**
- **Using MATLAB to compute the transfer function**

Pendulum





$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta - c\dot{\theta} + T(t)$$

$$x_1 \triangleq \theta,\ x_2 \triangleq \dot{\theta} \Rightarrow$$

$$\dot{x} = f(x,u)$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(g/\ell)\sin x_1 - (c/m\ell^2)x_2 + (1/m\ell^2)T(t) \end{aligned}$$

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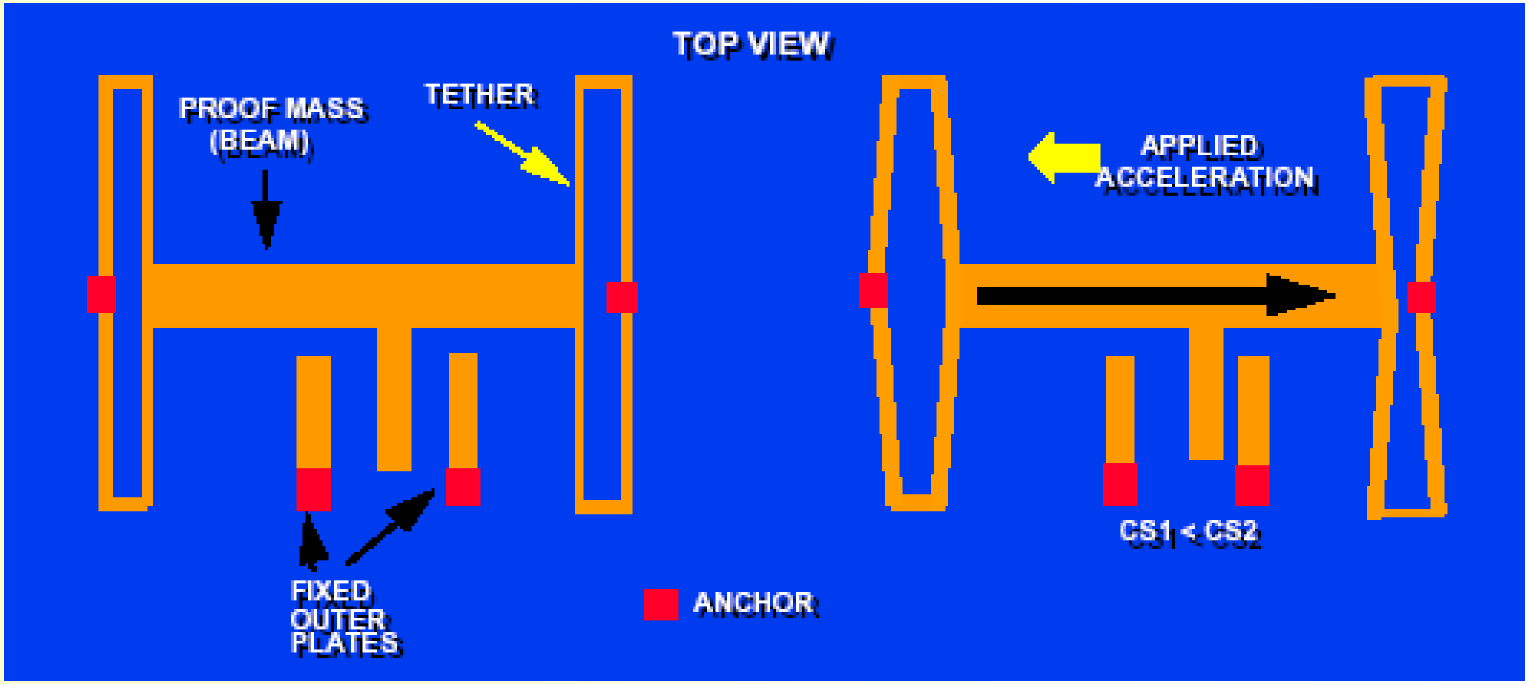
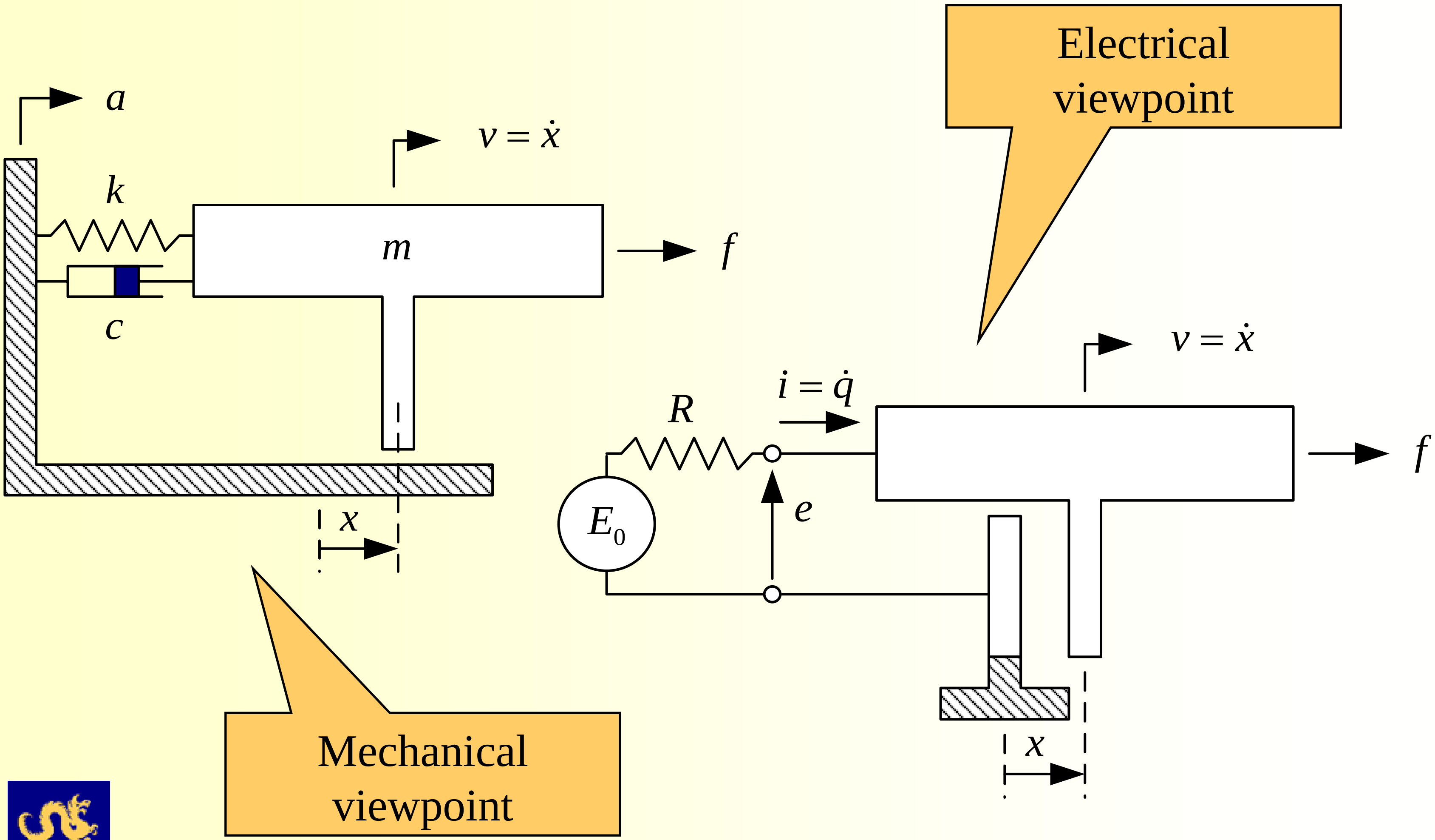
Micro-machined Capacitive Accelerometer



From James Doscher, Analog Devices

Accelerometer Model



Governing Equations

$$m(\ddot{x}+a) = f - kx - c\dot{x} \Rightarrow m\ddot{x} = f - kx - c\dot{x} - ma \quad \left(\sum \text{Forces on } m\right)$$

$$E_0 = Ri + e \Rightarrow R\dot{q} + \frac{q}{C(x)} = E_0 \quad \left(\sum \text{Voltages around loop}\right)$$

$$f = -\frac{q^2}{2C(x)^2}\frac{dC(x)}{dx}, \quad C(x) = C_0\frac{d_0}{d_0+x} \Rightarrow -\frac{1}{C(x)^2}\frac{dC(x)}{dx} = \frac{1}{C_0 d_0}$$

$$\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -\frac{q^2}{2m}\frac{1}{C_0 d_0} - \frac{k}{m}x - \frac{c}{m}v - a(t) \\
\dot{q} &= -\frac{q(d_0+x)}{RC_0 d_0} + \frac{1}{R}E_0
\end{aligned}$$

First order system of ODEs
- independent variable: t
- states: x, v, q (dependent variables)
- input: $a(t)$
- parameters: m, c, k, d_0, R, C_0, E_0

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Remarks on Capacitor

Electric power supplied to capacitor: ei

Mechanical power supplied to capacitor: fv

Total work on capacitor: $W = ei\,dt + fv\,dt = e\,dq + f\,dx$

Constitutive relation (electrically linear): $q = C(x)e$

The electric energy stored in capacitor is defined by

$$dE_e(q,x) = e\,dq + f\,dx \Rightarrow$$

$$E_e = \int_0^q \frac{q}{C(x)}dq = \frac{q^2}{2C(x)}$$

$$e = \frac{\partial E_e}{\partial q} = \frac{q}{C(x)},\ f = \frac{\partial E_e}{\partial x} = -\frac{q^2}{2C(x)^2}\frac{dC(x)}{dx}$$

Analyzing Ordinary Differential Equations

State vector

Input

Initial condition

system of first order odes: $\dot{x} = f\left(x, u\left(t\right)\right),\ x\left(t_0\right) = x_0$

- Steady-state (equilibrium) behavior
  - Set input equal to a constant
  - Set state time derivative (velocity) equal to zero
- Dynamic behavior
  - Time response to initial conditions
  - Time response to inputs

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Equilibrium Behavior

$$0 = v$$

$$0 = -\frac{q^2}{2m}\frac{1}{C_0 d_0} - \frac{k}{m}x - \frac{c}{m}v - \bar{a}$$

$$0 = -\frac{q\left(d_0 + x\right)}{RC_0 d_0} + \frac{1}{R}E_0$$

$$\Rightarrow$$

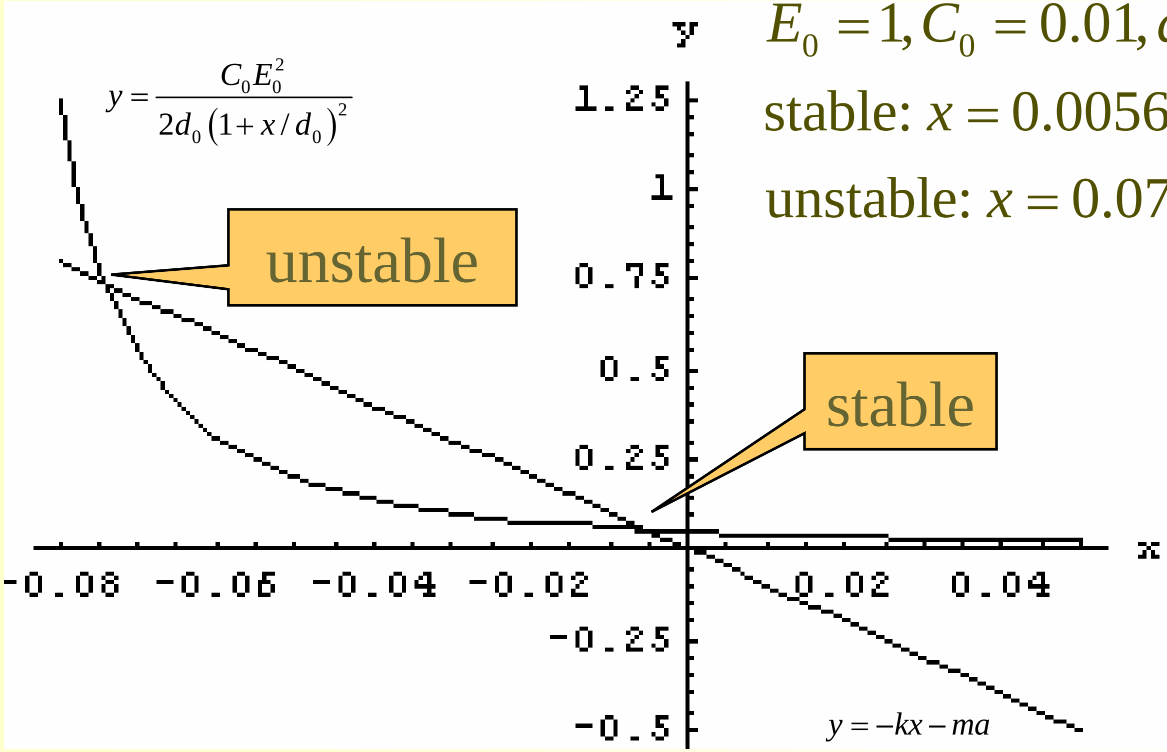
$$0 = -\frac{q^2}{2C_0 d_0} - kx - m\bar{a}$$

$$0 = -\frac{q\left(d_0 + x\right)}{C_0 d_0} + E_0$$

eliminate q

$$-kx - m\bar{a} = \frac{C_0 E_0^2}{2d_0\left(1 + x/d_0\right)^2}$$

Equilibrium Behavior



$E_0 = 1, C_0 = 0.01, d_0 = 0.1, k = 0$

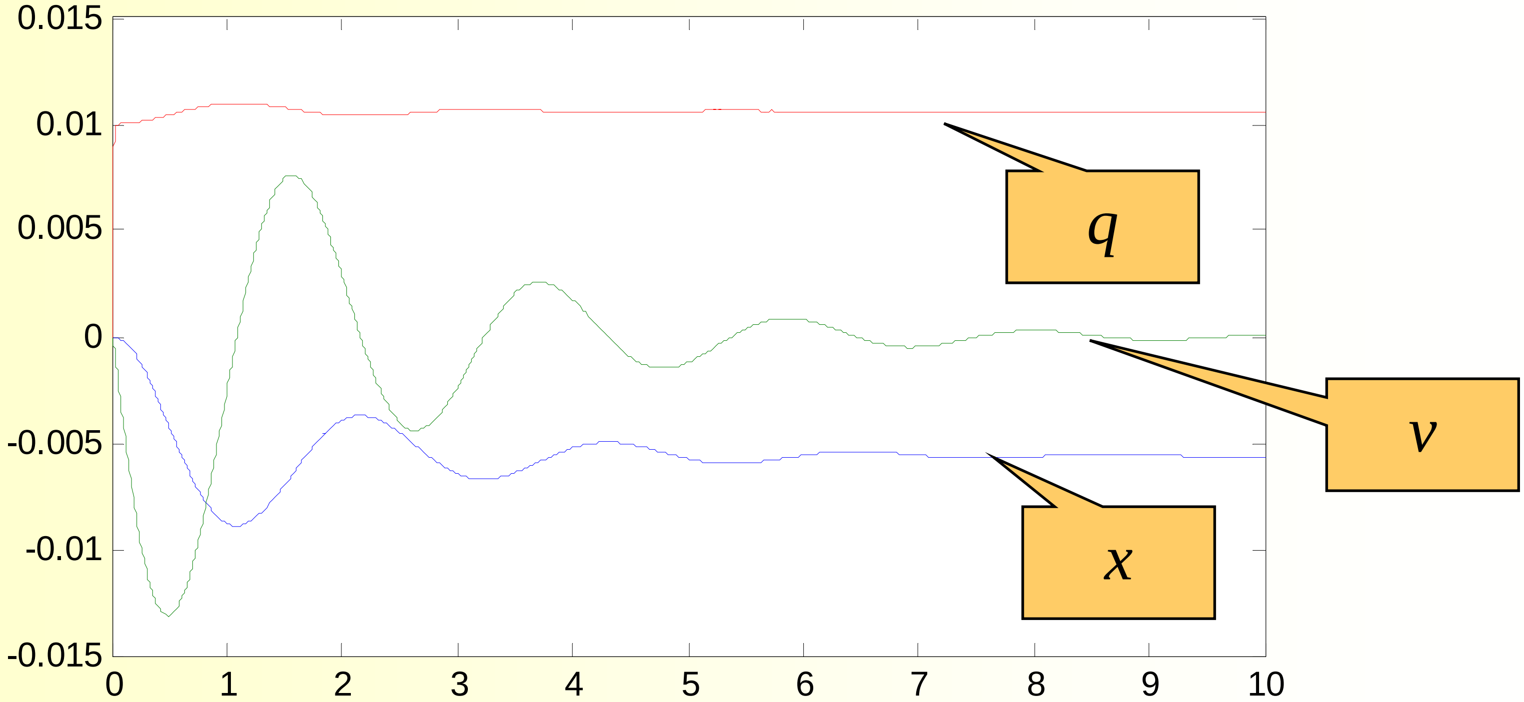
stable: $x = 0.005612, q = 0.010595$

unstable: $x = 0.074008, q = 0.038473$

Dynamic Behavior

Transient response from the origin: $x(0)=0, v(0)=0, q(0)=0$

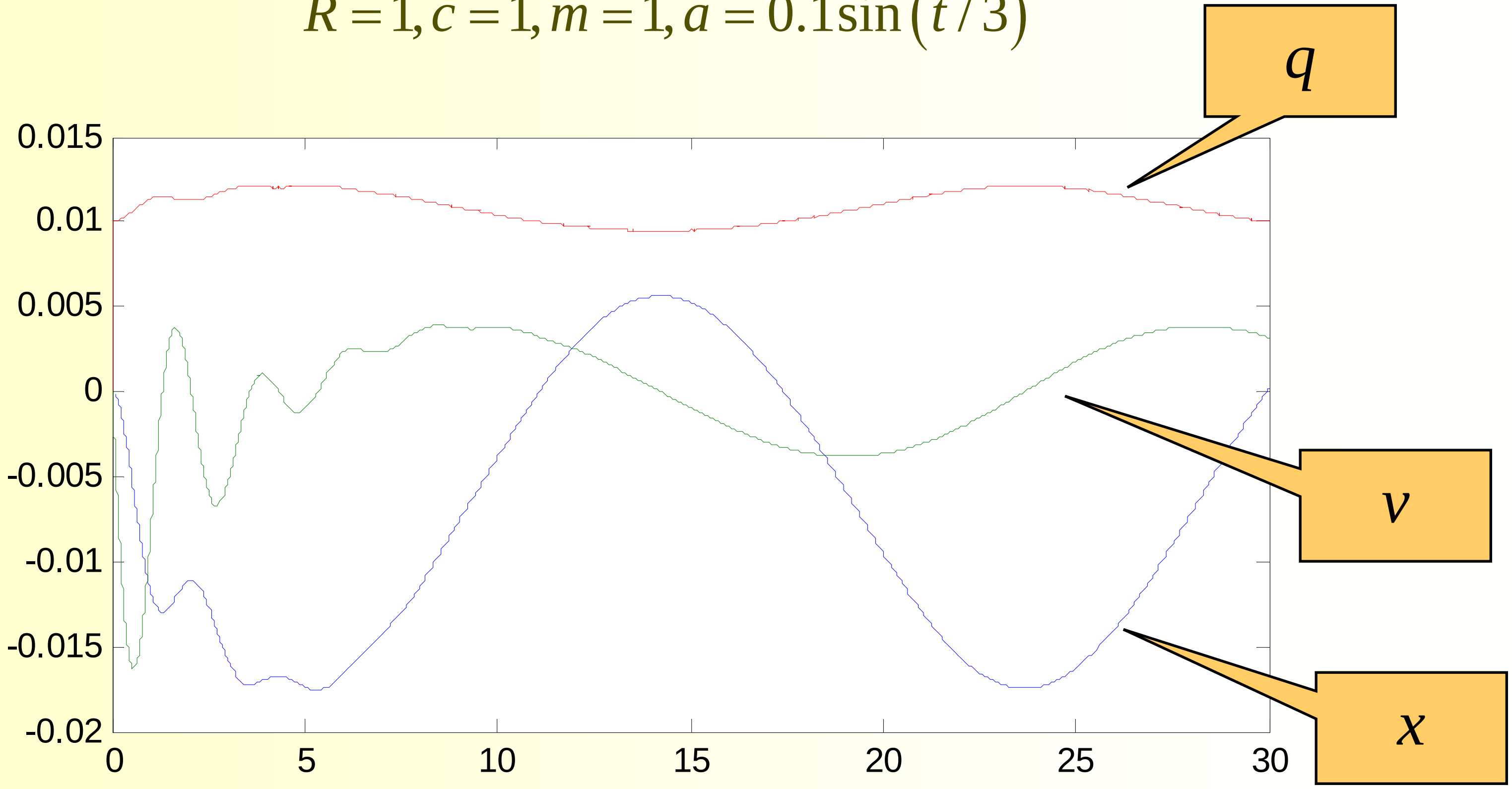
$$R=1, c=1, m=1, a=0$$



Dynamic Behavior

Transient response from the origin: $x(0)=0, v(0)=0, q(0)=0$

$$R=1, c=1, m=1, a=0.1\sin(t/3)$$



Solving ODEs

```
function dy = accel(t,y)
dy = zeros(3,1);    % a column vector
a=0;
dy(1) = y(2);
dy(2) = -y(3)^2/0.002-10*y(1)-1*y(2)-a;
dy(3) = -y(3) * (0.1+y(1))/0.001+1;
```

Write a function that defines the differential equations

```
>> options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-5]);
>> [T,Y] = ode45(@accel,[0 10],[0 0 0],options);
>> plot(T,Y(:,1),T,Y(:,2),T,Y(:,3))
```

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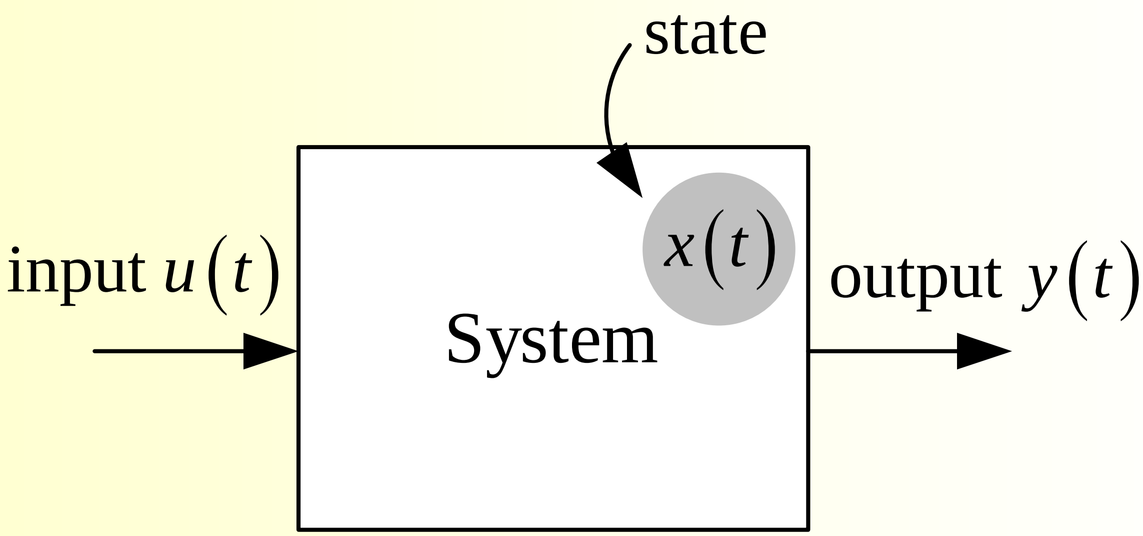
General Model of Nonlinear System



$$\dot{x} = f(x,u) \quad \text{state equation}$$

$$y = h(x,u) \quad \text{output equation}$$

$$x \in R^n, u \in R^m, y \in R^q$$

Linearization 1

In general models derived from physical principles involve sets of first and second order differential equations. These are reduced to first order standard form, involving the state vector *x*, input vector *u*, and output vector *y*.

$$\dot{x} = f(x,u) \quad \text{state equations}$$

$$y = h(x,u) \quad \text{output equations}$$

A set of values (x_0, u_0, y_0) is called and equilibrium point if they satisfy:

$$0 = f(x_0, u_0)$$

$$y_0 = h(x_0, u_0)$$

We are interested in motions that remain close to the equilibrium point.

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Linearization 2

Define: $x(t) = x_0 + \delta x(t), u(t) = u_0 + \delta u(t), y(t) = y_0 + \delta y(t)$

The equations become:

$$\delta \dot{x} = f\left(x_0 + \delta x(t), u_0 + \delta u(t)\right)$$
$$y_0 + \delta y(t) = h\left(x_0 + \delta x(t), u_0 + \delta u(t)\right)$$

Now, construct a Taylor series for f, h

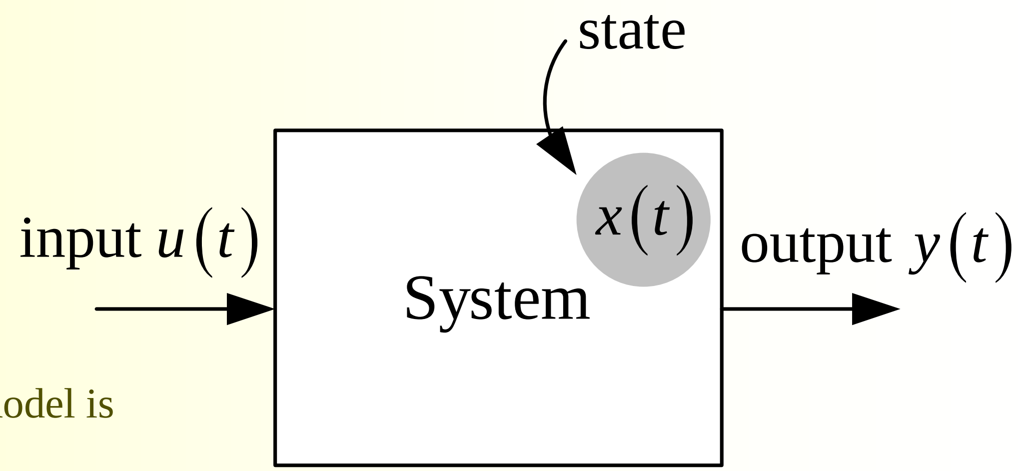
$$f\left(x_0 + \delta x, u_0 + \delta u\right) = f\left(x_0, u_0\right) + \frac{\partial f\left(x_0, u_0\right)}{\partial x}\delta x + \frac{\partial f\left(x_0, u_0\right)}{\partial u}\delta u + hot$$

$$h\left(x_0 + \delta x, u_0 + \delta u\right) = h\left(x_0, u_0\right) + \frac{\partial h\left(x_0, u_0\right)}{\partial x}\delta x + \frac{\partial h\left(x_0, u_0\right)}{\partial u}\delta u + hot$$

Notice that $f\left(x_0, u_0\right) = 0$ and $h\left(x_0, u_0\right) = y_0$, so

$$\begin{aligned} \delta \dot{x} &= \frac{\partial f\left(x_0, u_0\right)}{\partial x}\delta x + \frac{\partial f\left(x_0, u_0\right)}{\partial u}\delta u \\ \delta y &= \frac{\partial h\left(x_0, u_0\right)}{\partial x}\delta x + \frac{\partial h\left(x_0, u_0\right)}{\partial u}\delta u \end{aligned} \Rightarrow \boxed{\begin{aligned} \delta \dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u \end{aligned}}$$

Linear System Models: state space & transfer function



The diffrential equation or 'state space' model is

$\dot{x}(t) = Ax(t) + Bu(t)$ state equation

$y(t) = Cx(t) + Du(t)$ output equation

$x(0) = x_0$ initial condition

The state space model describes how the input $u(t)$ and the initial condition affect the state $x(t)$ and the output $y(t)$. The 'transfer function' model is

$Y(s) = G(s)U(s)$

where $Y(s), U(s)$ are the Laplace transforms of the output and input and $G(s)$ is the transfer function. The transfer function model describes how the input affects the output.

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Solving ODEs via the Laplace Transform

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$\begin{aligned} \mathcal{L}(\dot{x}) &= A\mathcal{L}(x) + B\mathcal{L}(u) \\ \mathcal{L}(y) &= C\mathcal{L}(x) + D\mathcal{L}(u) \end{aligned} \quad \Rightarrow \quad \begin{aligned} sX(s) - x_0 &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$X(s) = [sI - A]^{-1} x_0 + [sI - A]^{-1} BU(s)$$

$$Y(s) = C[sI - A]^{-1} x_0 + \left\{ C[sI - A]^{-1} B + D \right\} U(s)$$

Initial condition response

Input response

Transfer function: $\boxed{G(s) = C[sI - A]^{-1} B + D}$

Example: Accelerometer

$$\dot{x} = v$$

$$\dot{v} = -\frac{q^2}{2m}\frac{1}{C_0 d_0} - \frac{k}{m}x - \frac{c}{m}v - a(t) \qquad e = -\frac{q(d_0 + x)}{C_0 d_0}$$

$$\dot{q} = -\frac{q(d_0 + x)}{RC_0 d_0} + \frac{1}{R}E_0$$

$$\delta\dot{x} = \delta v$$

$$\delta\dot{v} = -\frac{q_0}{m}\frac{1}{C_0 d_0}\delta q - \frac{k}{m}\delta x - \frac{c}{m}\delta v - \delta a(t)$$

$$\dot{q} = -\frac{(d_0 + x_0)}{RC_0 d_0}\delta q - \frac{q_0}{RC_0 d_0}\delta x$$

$$\delta e = -\frac{(d_0 + x_0)}{C_0 d_0}\delta q - \frac{q_0}{C_0 d_0}\delta x$$

Example: Accelerometer

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{v} \\ \delta\dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{k}{m} & -\dfrac{c}{m} & -\dfrac{q_0}{m}\dfrac{1}{C_0 d_0} \\ -\dfrac{q_0}{RC_0 d_0} & 0 & -\dfrac{(d_0 + x_0)}{RC_0 d_0} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta v \\ \delta q \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \delta a$$

$$\delta e = \begin{bmatrix} -\dfrac{q_0}{C_0 d_0} & 0 & -\dfrac{(d_0 + x_0)}{C_0 d_0} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta v \\ \delta q \end{bmatrix} + [0]\delta a$$

state: $x = [\delta x, \delta v, \delta q]^T$

input: $u = [\delta a]$ (acceleration)

output: $y = [\delta e]$ (capacitor voltage)

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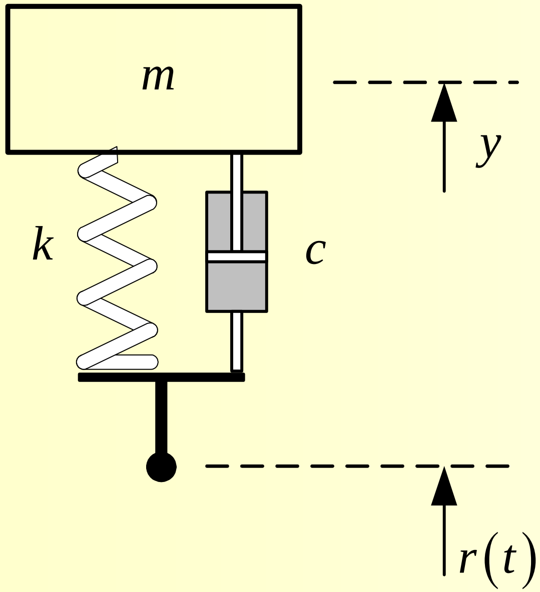
Accelerometer Transfer Function

```
>> A=[0 1 0;-10 -0.1 -(0.010595)/(0.01*0.1);-
(0.010595)/(0.01*0.1) 0 -(0.1+0.005612)/(0.01*0.1)];
>> B=[0;-1;0];
>> C=[-(0.010595)/(0.01*0.1) 0 -(0.1+0.005612)/(0.01*0.1)];
>> Gss=ss(A,B,C,0);
>> G=tf(Gss)
Transfer function:
       10.6 s + 3.367e-013
---------------------------------
s^3 + 105.7 s^2 + 20.56 s + 943.9
>> zpk(G)
Zero/pole/gain:
             10.595 s
----------------------------------
(s+105.6) (s^2  + 0.1101s + 8.938)
```

Reduction to State Space Form

$$m\ddot{y} = -k\left(y - r\right) - c\left(\dot{y} - \dot{r}\right)$$



Try the usual: $x_1 = y$, $x_2 = \dot{x}_1 = \dot{y}$ $\Rightarrow$ $\dot{x}_1 = x_2$, $\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{k}{m}r + \frac{c}{m}\dot{r}$

An alternative: $x_1 = y - \alpha_1 r$, $x_2 = \dot{x}_1 - \alpha_2 r$ $\Rightarrow$ $y = x_1 + \alpha_1 r$, $\dot{y} = x_2 + \alpha_1\dot{r} + \alpha_2 r$

$$\dot{x}_1 = x_2 + \alpha_2 r$$

$$\dot{x}_2 + \alpha_1\ddot{r} + \alpha_2\dot{r} = -\frac{k}{m}\left(x_1 + \alpha_1 r\right) - \frac{c}{m}\left(x_2 + \alpha_1\dot{r} + \alpha_2 r\right) + \frac{k}{m}r + \frac{c}{m}\dot{r}$$

$$\alpha_1 = 0 \text{ and } \alpha_2 = -\frac{c}{m}\alpha_1 + \frac{c}{m} \Rightarrow \alpha_1 = 0, \alpha_2 = \frac{c}{m}$$

$$\dot{x}_1 = x_2 + \frac{c}{m}r$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \left(-\left(\frac{c}{m}\right)^2 + \frac{k}{m}\right)r$$

Computing Transfer Functions

From original equation:

$$m\ddot{y} = -k\left(y - r\right) - c\left(\dot{y} - \dot{r}\right)$$

Use $L\left[\dot{y}\right] = sL\left[y\right] - y\left(0\right) \Rightarrow L\left[\ddot{y}\right] = sL\left[\dot{y}\right] - \dot{y}\left(0\right) = s^2 L\left[y\right] - sy\left(0\right) - y\left(0\right)$

with $y\left(0\right)$=0 and $\dot{y}\left(0\right) = 0$

$$ms^2 Y\left(s\right) = -kY\left(s\right) - csY\left(s\right) + kR\left(s\right) + csR\left(s\right)$$

$$Y\left(s\right) = \frac{cs + k}{ms^2 + cs + k} R\left(s\right) \Rightarrow G\left(s\right) = \frac{cs + k}{ms^2 + cs + k}$$

From state space

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c/m \\ \left(c/m\right)^2 + k/m \end{bmatrix} r, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} r$$

$$G\left(s\right) = C\left[sI - A\right]^{-1} B + D = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ k/m & s + -c/m \end{bmatrix}^{-1}\begin{bmatrix} c/m \\ -\left(c/m\right)^2 + k/m \end{bmatrix} = \frac{cs + k}{ms^2 + cs + k}$$

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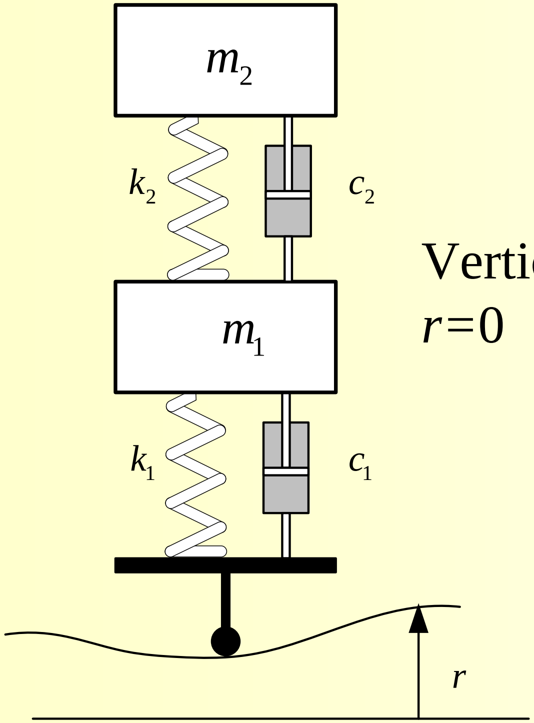
Quarter-Car Suspension



$$m_1\ddot{y}_1 = -k_2\left(y_1 - y_2\right) - k_1\left(y_1 - r\right) - c_2\left(\dot{y}_1 - \dot{y}_2\right) - c_1\left(\dot{y}_1 - \dot{r}\right)$$

$$m_2\ddot{y}_2 = k_2\left(y_1 - y_2\right) + c_2\left(\dot{y}_1 - \dot{y}_2\right)$$

Verticle motion, y_1, y_2 measured from rest with $r=0$

Using Laplace Transform, derive a relationship between $R(s)$ and $Y_2(s)$ (data from Franklin et al)

$$Y_2(s) = \frac{(17.333/1.3067)s}{s^4 + 490s^3 + 44067.5s^2 + 157061s + 1.88099} R(s)$$

Quarter-Car State Space Model

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{c_1}{m_1} \\ -(c_1+c_2)\frac{c_1}{m_1} + \frac{k_1}{m_1} \\ 0 \\ \frac{c_2 c_1}{m_1} \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

Summary

- Accelerometer model
  - Start with physical principles
  - Nonlinear ordinary differential equations
  - First order form: ‘standard form’ or ‘state varaiable form’
- State space and transfer functions
- Solving ODEs / Transfer Functions
  - Linear systems are described by state space and transfer function models
  - Linear systems of ODEs can be solved using Laplace transforms
  - The transfer function is used to describe the input-output response
- Using MATLAB to solve odes
- Using MATLAB to compute transfer functions

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