MEM 255 Introduction to Control Systems: *Analyzing Dynamic Response*

Harry G. Kwatny

Department of Mechanical Engineering & Mechanics

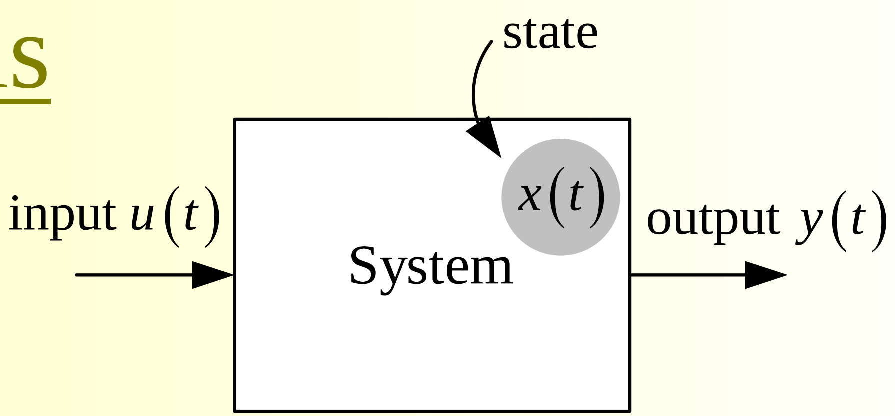
Drexel University

Outline

- **Time domain and frequency domain**
- **A second order system**
  - **Via partial fraction expansion**
  - **Using Matlab**
- **Time response parameters**
- **Frequency response & Bode plots**
- **Example: Suspension System**

Drexel UNIVERSITY

Time Domain vs Frequency Domain Analysis



Time domain: take $u(t)$ to be a 'simple' function usually an impulse or a step. Characterize the output response, $y(t)$, in terms of the time response graph.

Frequency response: Take $u(t)$ to be the sinusoid $\sin \omega t$. Characterize the output response, $y(t) = B \sin(\omega t + \phi)$, in terms of $B(\omega), \phi(\omega)$.

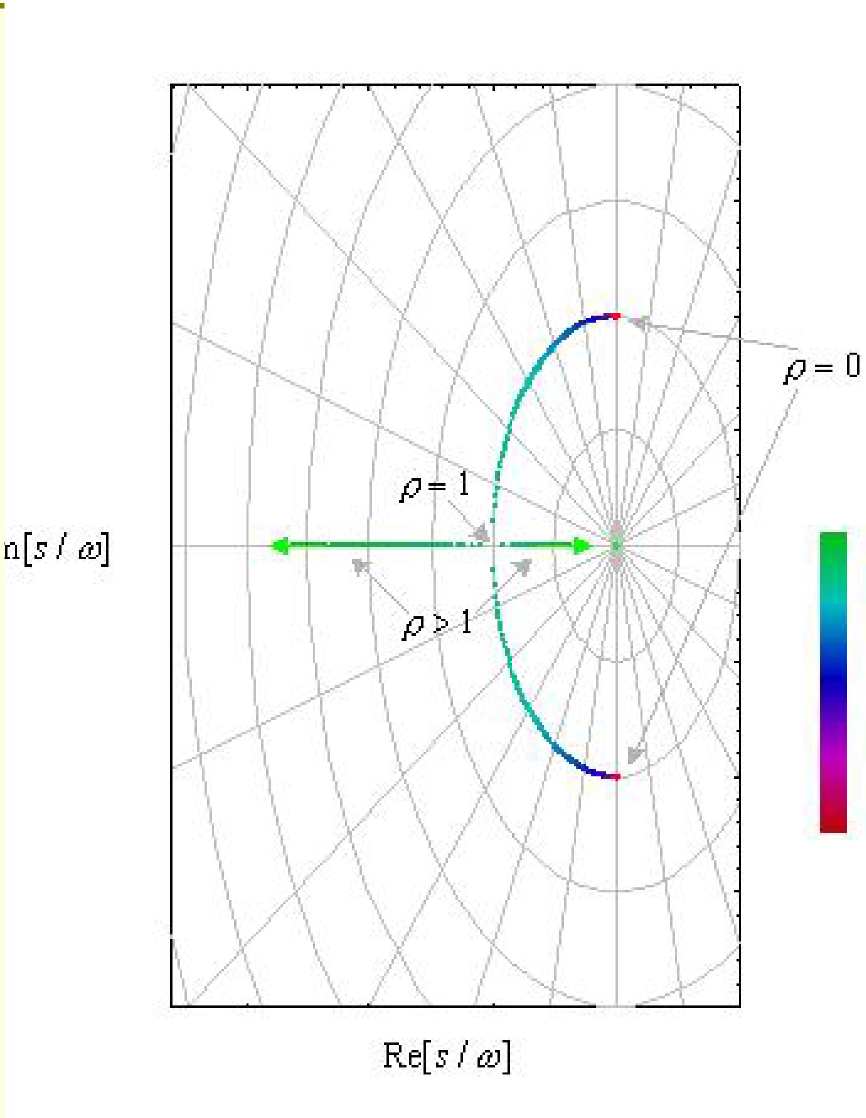
2^{nd} Order System-Factoring the Denominator

$$G(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + b_1 s + b_0}$$

$\Downarrow$ special case

$$G(s) = \frac{\omega_n^2}{s^2 + 2\rho s + \omega_n^2}$$

$$= \frac{\omega_n^2}{\left(s + \omega\rho \pm \omega\sqrt{-1+\rho}\right)}$$



Drexel UNIVERSITY

Step Response of 2nd Order System

$$Y(s)=\frac{\omega_n^2}{s^2+2\rho s+\omega_n^2}U(s)=\frac{\omega_n^2}{s^2+2\rho s+\omega_n^2}\frac{1}{s}$$

$$=\frac{\omega_n^2}{\left(s+\omega_n\rho+\omega_n\sqrt{-1+\rho}\right)\left(s+\omega_n\rho-\omega_n\sqrt{-1+\rho}\right)s}$$

$$=\frac{c_1}{\left(s+\omega_n\rho+\omega_n\sqrt{-1+\rho}\right)}+\frac{c_2}{\left(s+\omega_n\rho-\omega_n\sqrt{-1+\rho}\right)}+\frac{c_3}{s}$$

$$c_1=\lim_{s\to-\omega_n\rho-\omega_n\sqrt{-1+\rho}}\frac{\left(s+\omega_n\rho+\omega_n\sqrt{-1+\rho}\right)\omega_n^2}{\left(s+\omega_n\rho+\omega_n\sqrt{-1+\rho}\right)\left(s+\omega_n\rho-\omega_n\sqrt{-1+\rho}\right)s}$$

$$c_1=\frac{-1}{2\sqrt{-1+\rho}\left(\rho+\sqrt{-1+\rho}\right)},\quad c_2=\frac{1}{2\sqrt{-1+\rho}\left(\rho-\sqrt{-1+\rho}\right)}$$

$$c_3=\lim_{s\to 0}\frac{s\omega_n^2}{s^2+2\rho s+\omega_n^2}\frac{1}{s}=1$$

Step Response of 2nd Order System

$$y(t) = L^{-1}\left[Y(s)\right] = c_1 L^{-1}\left[\frac{1}{\left(s+\omega_n\rho+\omega_n\sqrt{-1+\rho}\right)}\right]$$

$$+c_2 L^{-1}\left[\frac{1}{\left(s+\omega_n\rho-\omega_n\sqrt{-1+\rho}\right)}\right]+c_3 L^{-1}\left[\frac{1}{s}\right]$$

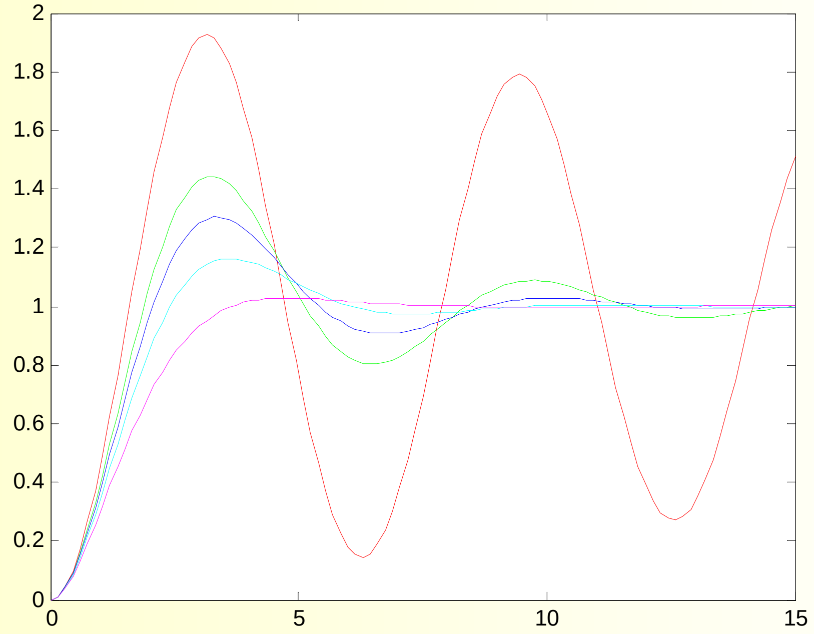
$$= c_1 u_s(t) e^{-t\omega_n\left(\rho+\sqrt{-1+\rho^2}\right)} + c_2 u_s(t) e^{-t\omega_n\left(\rho-\sqrt{-1+\rho^2}\right)} + u_s(t)$$

For $\rho < 1,$ the roots are complex

$$y(t) = 1 - e^{-t\omega_n\rho}\left(\cos\left[t\omega_n\sqrt{1-\rho^2}\right] + \frac{\rho}{\sqrt{1-\rho^2}}\sin\left[t\omega_n\sqrt{1-\rho^2}\right]\right)$$

Step Response Using MATLAB

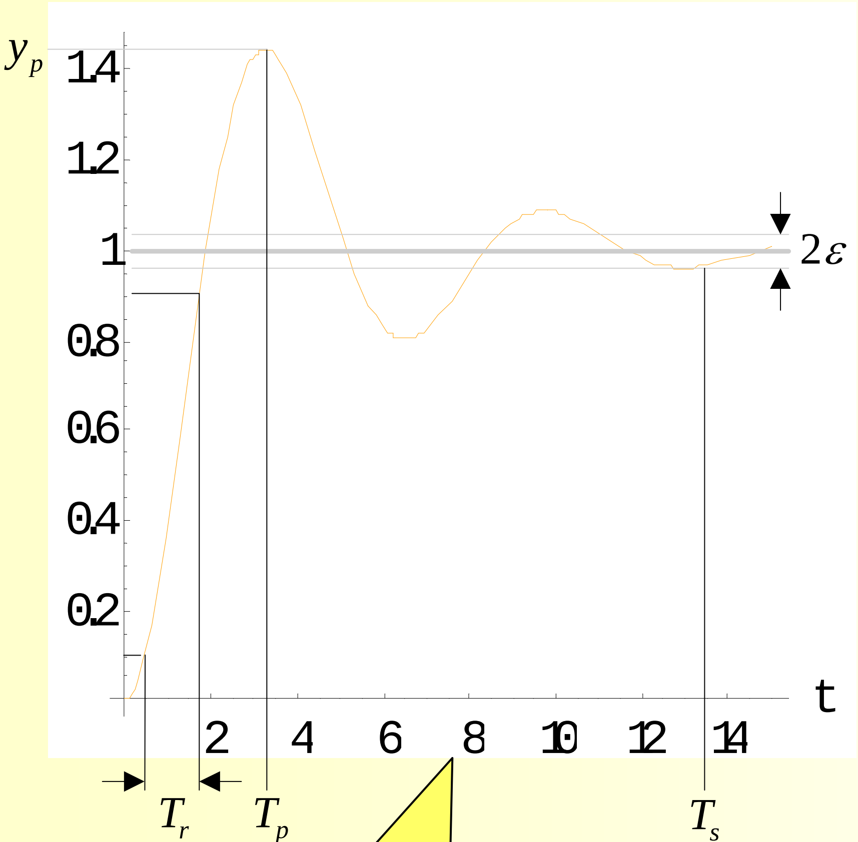
```
>>[Y1,T1]=step(1/(s^2+0.05*s+1),15);
>>[Y2,T2]=step(1/(s^2+0.5*s+1),15);
>>[Y3,T3]=step(1/(s^2+0.707*s+1),15);
>>[Y4,T4]=step(1/(s^2+1.0*s+1),15);
>>[Y5,T5]=step(1/(s^2+1.5*s+1),15);
>>plot(T1,Y1,'r',T2,Y2,'g',T3,Y3,'b',T4,Y4,'c',T5,Y5,'m')
```



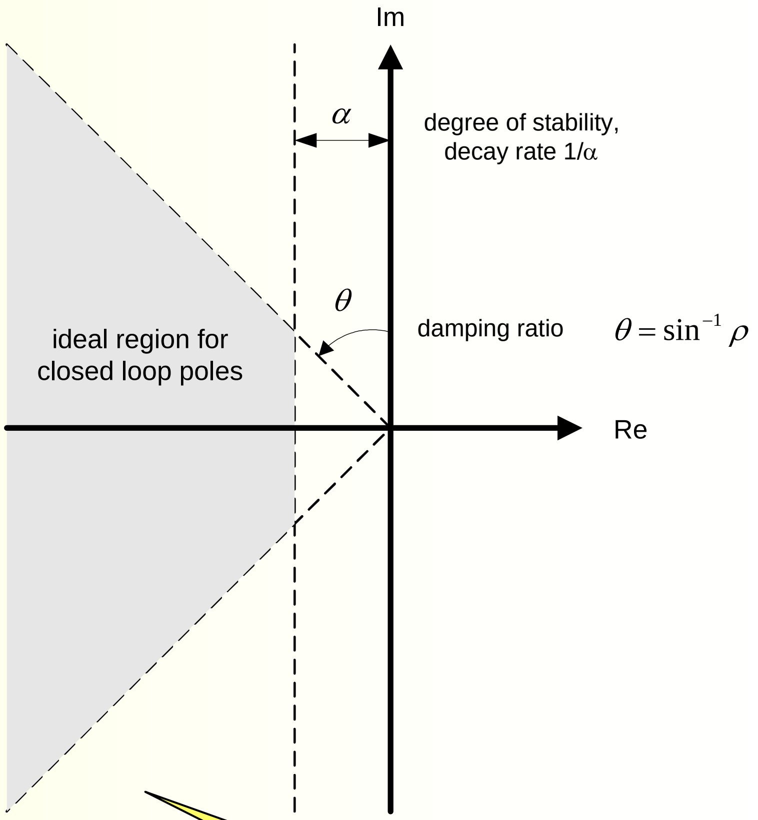
Traditional Parameters, 1

- *rise time*, T_r, usually defined as the time to get from 10% to 90% of its ultimate (i.e., final) value.
- *settling time*, T_s, the time at which the trajectory first enters an ε-tolerance of its ultimate value and remains there (ε is often taken as 2% of the ultimate value).
- *peak time*, T_p, the time at which the trajectory attains its peak value.
- *peak overshoot*, *OS*, the peak or supreme value of the trajectory ordinarily expressed as a percentage of the ultimate value of the trajectory. An overshoot of more than 30% is often considered undesirable. A system without overshoot is 'overdamped' and may be too slow (as measured by rise time and settling time).

Traditional Parameters, 2

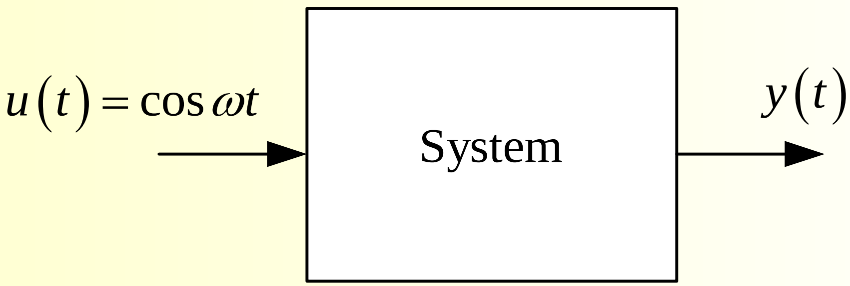


Time response parameters



Ideal pole locations

Frequency Response, 1



Trick: replace $\cos \omega t$ by $e^{j\omega t}$

recall $e^{j\omega t} = \cos \omega t + j \sin \omega t$

superposition $\Rightarrow$ if $\cos \omega t \to y_1(t), \sin \omega t \to y_2(t)$

then $e^{j\omega t} \to y_1(t) + jy_2(t)$

Frequency Response, 2

Suppose, $G(s) = K\dfrac{n(s)}{d(s)}$

$$Y(s) = G(s)\mathscr{L}\left[e^{j\omega t}\right] = K\frac{n(s)}{d(s)}\frac{1}{s-j\omega} = \frac{\tilde{n}(s)}{d(s)} + \frac{c_1}{s-j\omega}$$

$$c_1 = \lim_{s\to j\omega}\left\{(s-j\omega)K\frac{n(s)}{d(s)}\frac{1}{s-j\omega}\right\} = K\frac{n(j\omega)}{d(j\omega)} = G(j\omega)$$

$$Y(s) = \underbrace{\frac{\tilde{n}(s)}{d(s)}}_{transient\ response} + \underbrace{\frac{G(j\omega)}{s-j\omega}}_{steady-state\ response} \Leftrightarrow y(t) = y_{trans}(t) + y_{ss}(t)$$

Suppose $y_{trans}(t) \to 0$ with time, so $y(t) \to \boxed{y_{ss}(t) = G(j\omega)e^{j\omega t}}$

Frequency Response, 3

For each specified ω, $G(\omega)$ is a complex number, write in polar form:

$$G(\omega) = \left|G(j\omega)\right| e^{\angle G(j\omega)} \underset{\text{write as}}{\rightarrow} \rho(\omega) e^{j\phi(\omega)}$$

$$y_{ss}(t) = \rho(\omega) e^{j\phi(\omega)} e^{j\omega t} = \rho(\omega) e^{j(\omega t + \phi(\omega))}$$

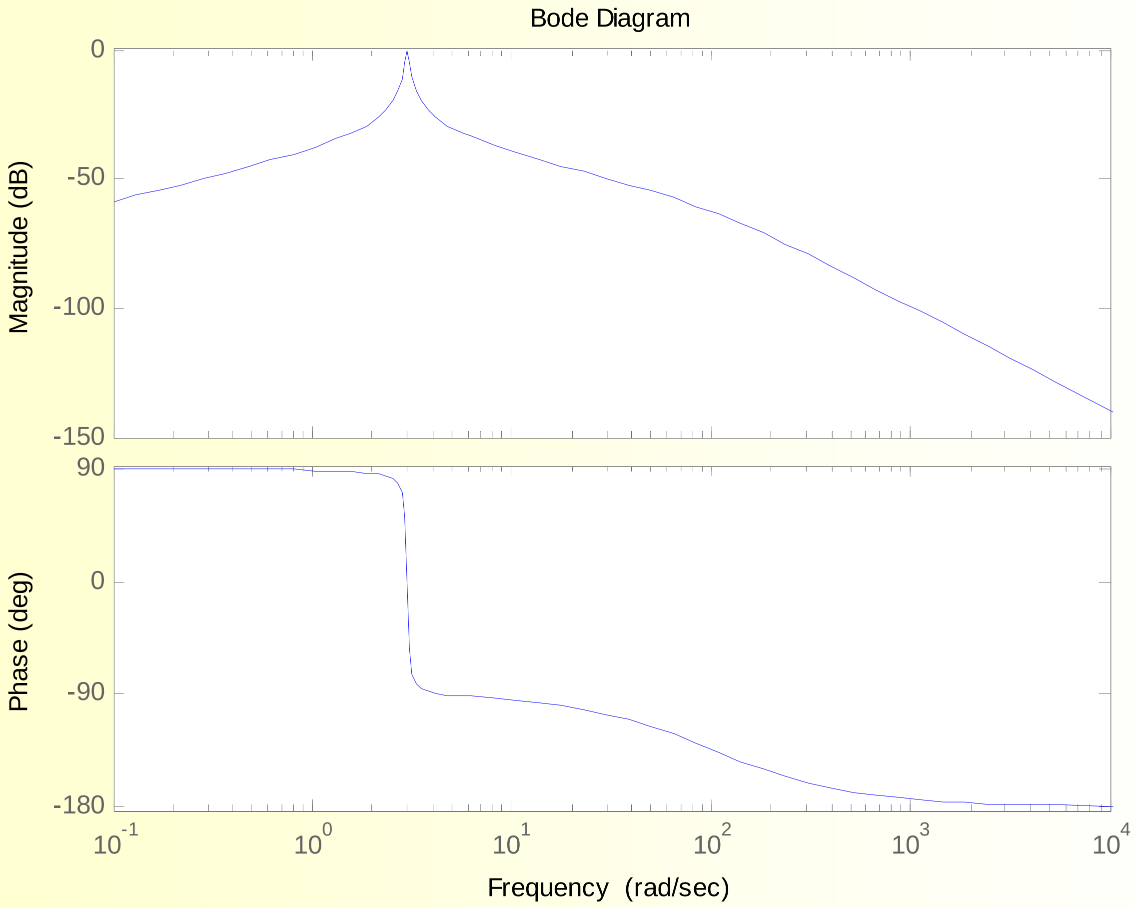
$$= \rho(\omega)\cos(\omega t + \phi(\omega)) + j\rho(\omega)\sin(\omega t + \phi(\omega))$$

$$\boxed{\begin{aligned} &\cos\omega t \rightarrow \rho(\omega)\cos(\omega t + \phi(\omega)) \\ &\sin\omega t \rightarrow \rho(\omega)\sin(\omega t + \phi(\omega)) \end{aligned}}$$

$G(j\omega) = \rho(\omega) e^{j\phi(\omega)}$ is called the frequency transfer function

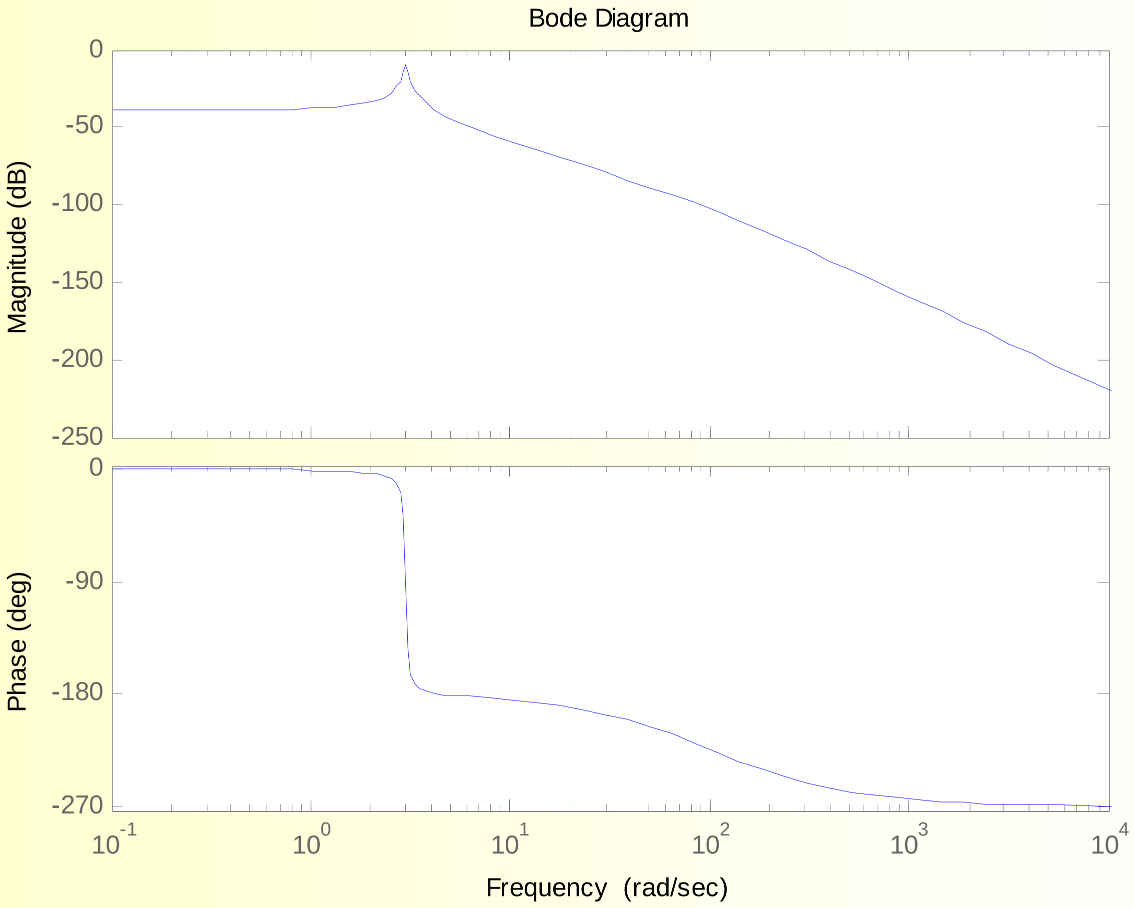
Accelerometer

```
Gss=ss(A,B,C,0);
G=tf(Gss);
bode(G)
```

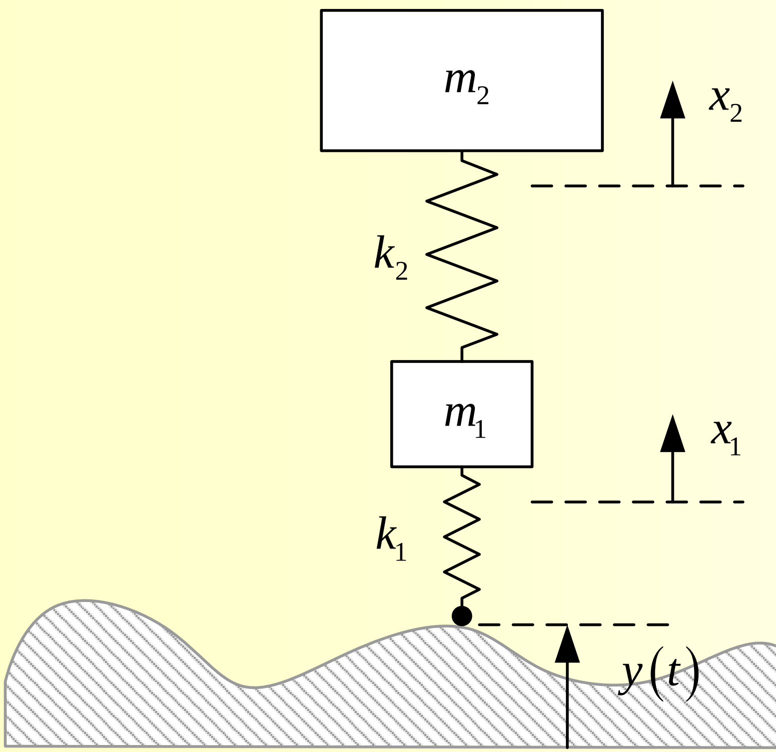


Accelerometer, Output Integrator Compensated

```
G1=G*(1/s);
bode(G1)
```



Auto Suspension: quarter car model



$$m_1\ddot{x}_1 = c\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) - k_1\left(x_1 - y(t)\right)$$

$$m_2\ddot{x}_2 = -k_2\left(x_2 - x_1\right) - c\left(\dot{x}_2 - \dot{x}_1\right)$$

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = -\frac{\left(k_1 + k_2\right)}{m_1}x_1 + \frac{k_2}{m_1}x_2 - \frac{c}{m_1}v_1 + \frac{c}{m_1}v_2 + \frac{k_1}{m_1}y(t)$$

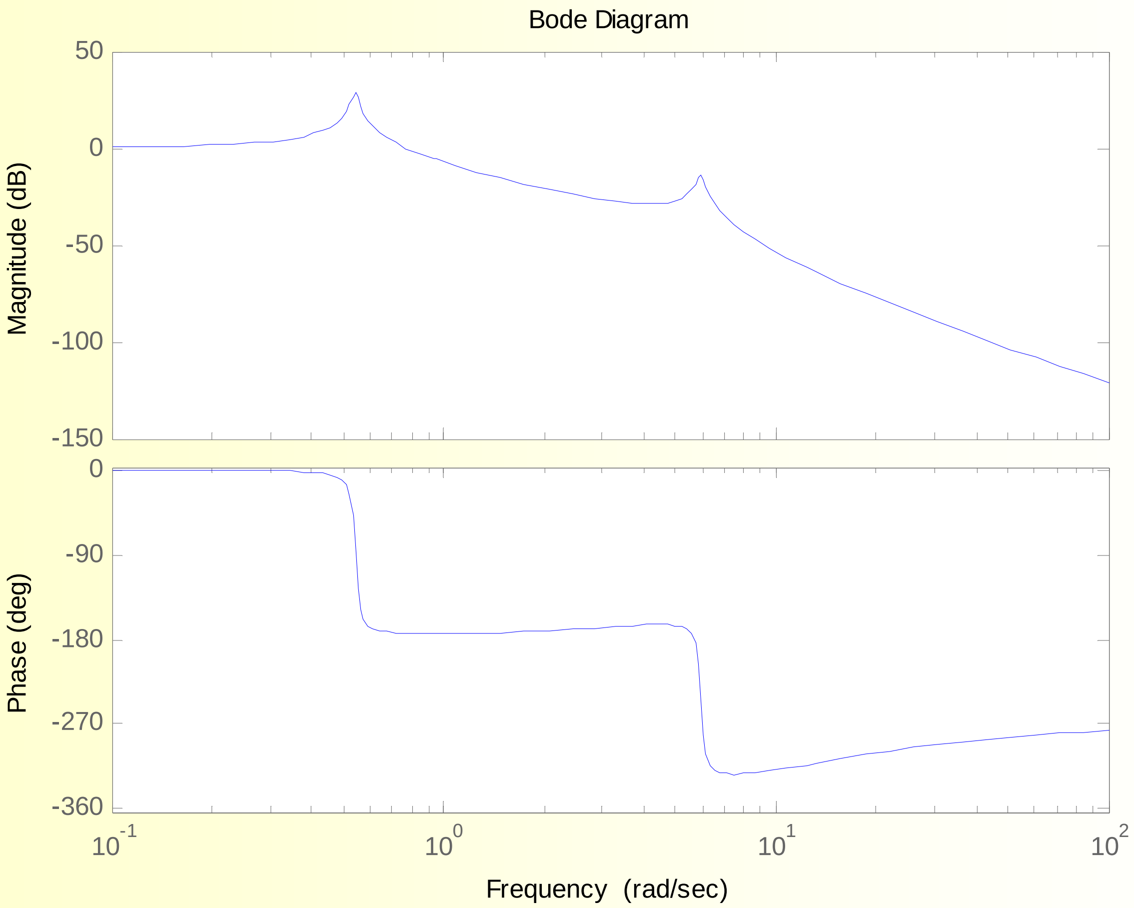
$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_2 + \frac{c}{m_2}v_1 - \frac{c}{m_2}v_2$$

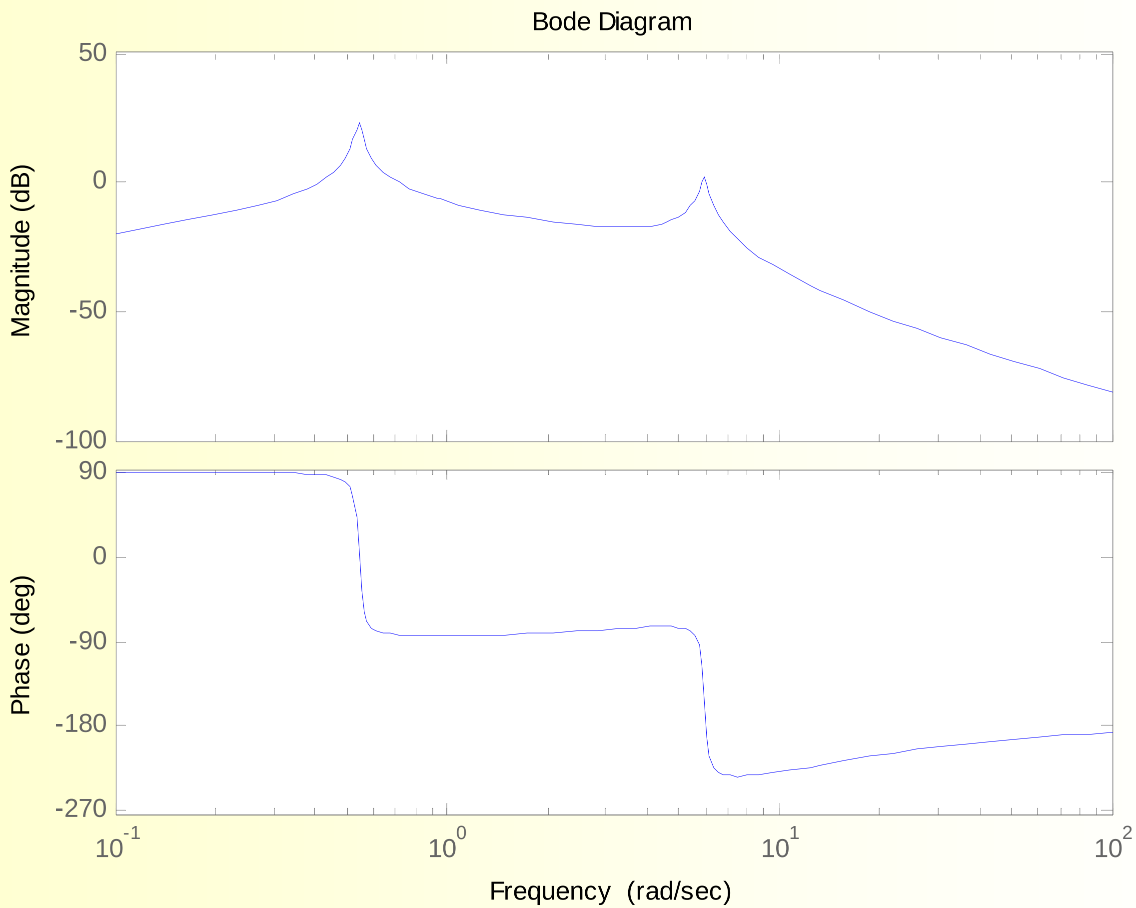
Suspension

```
g = 32.2;
m2 = 1000/g;
m1 = 100/g;
k2 = 10;
k1 = 10*k2;
c=2*0.707*sqrt(k2/m2)
A=[0 1 0 0;(-k1-k2)/m1,-c/m1,k2/m1,c/m1;0,0,0,1;k2/m2,c/m2,-k2/m2,-c/m2];
B=[0;k1/m1;0;0];
% payload position
C=[0 0 1 0];
Gss=ss(A,B,C,0)
bode(Gss,{0.1,100})
zpk(Gss)
% payload velocity
pause
C=[0 0 0 1];
Gss=ss(A,B,C,0)
bode(Gss,{0.1,100})
zpk(Gss)
% payload acceleration
pause
C=[k2/m2 c/m2 -k2/m2 -c/m2];
Gss=ss(A,B,C,0)
bode(Gss,{0.1,100})
zpk(Gss)
```

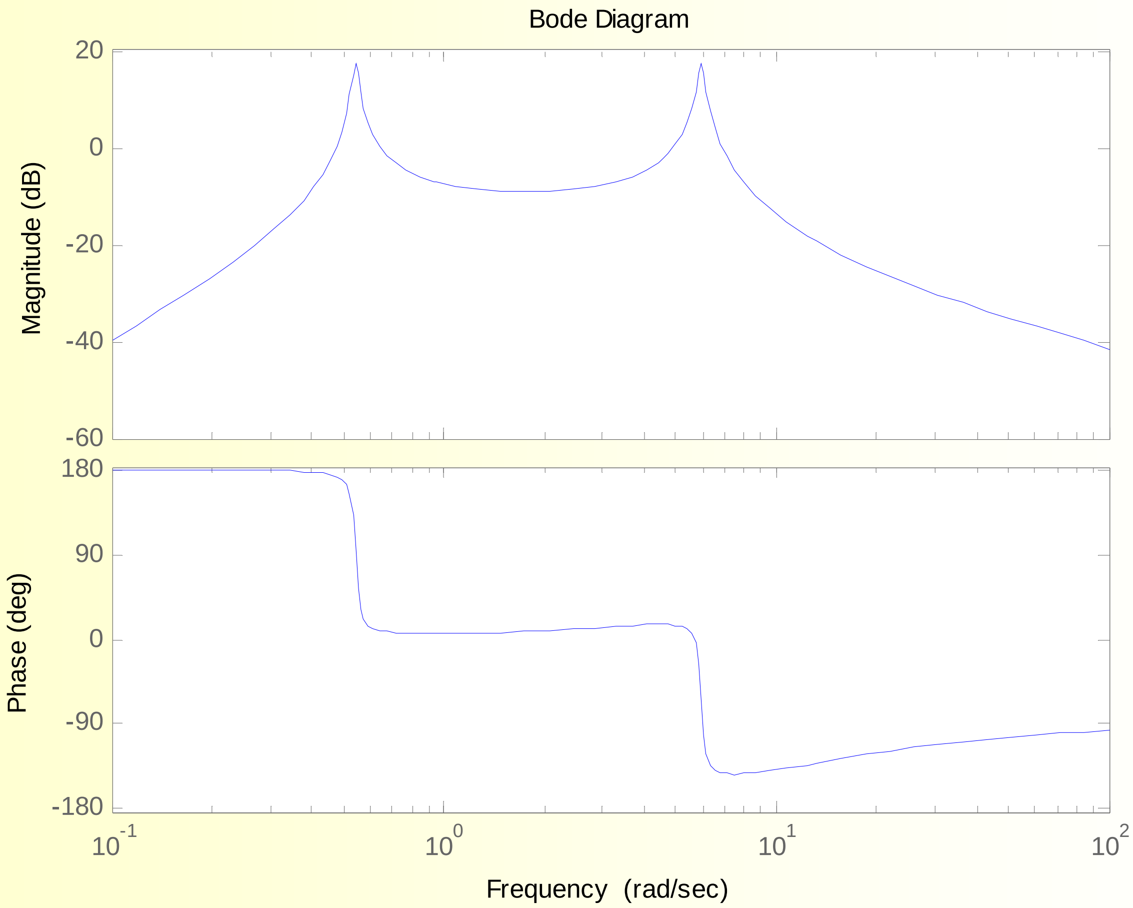
Suspension: displacement response



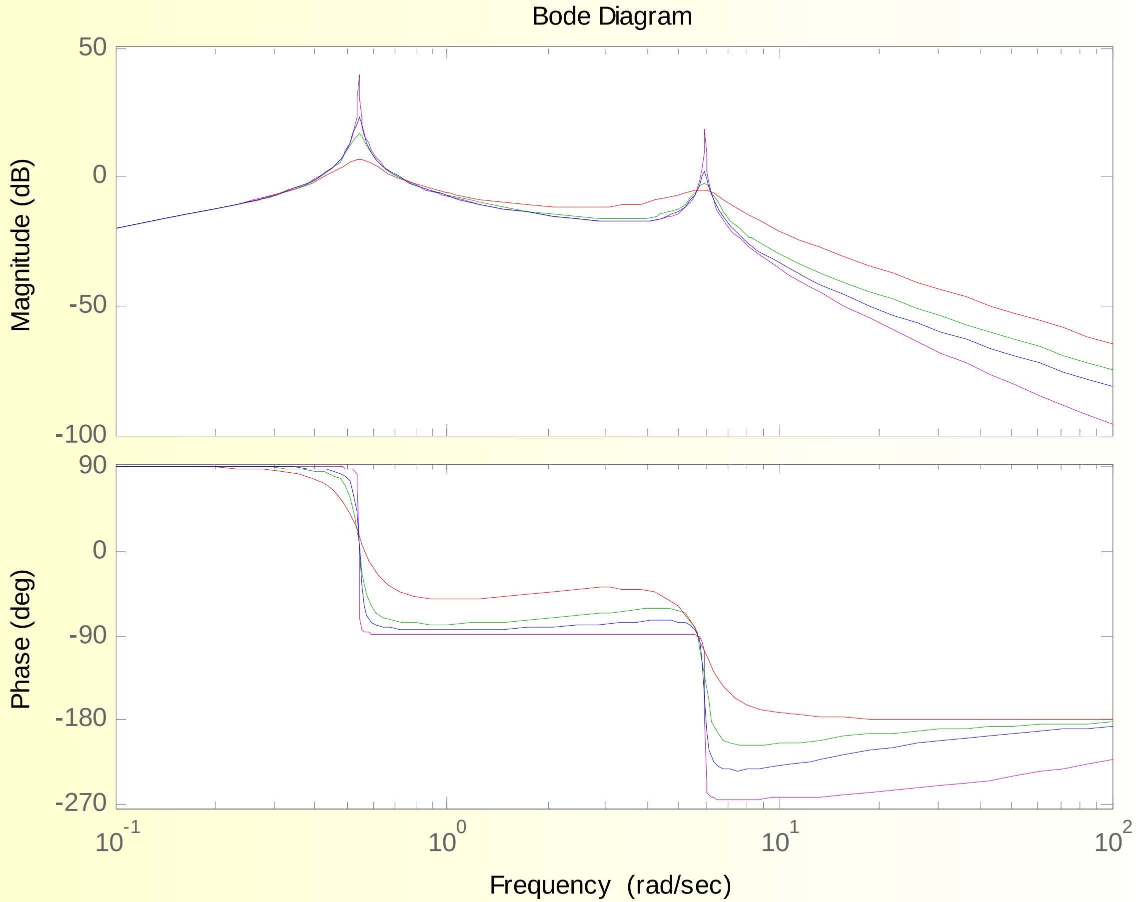
Suspension: velocity response



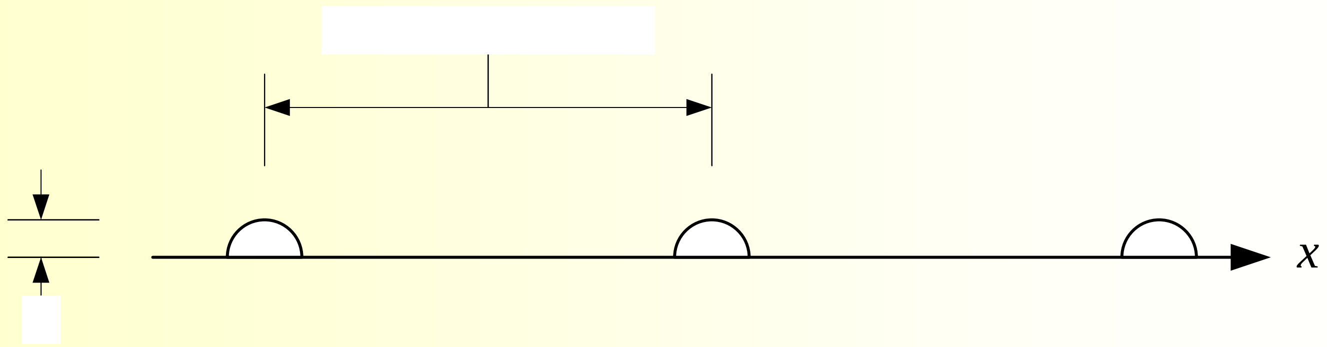
Suspension: acceleration response

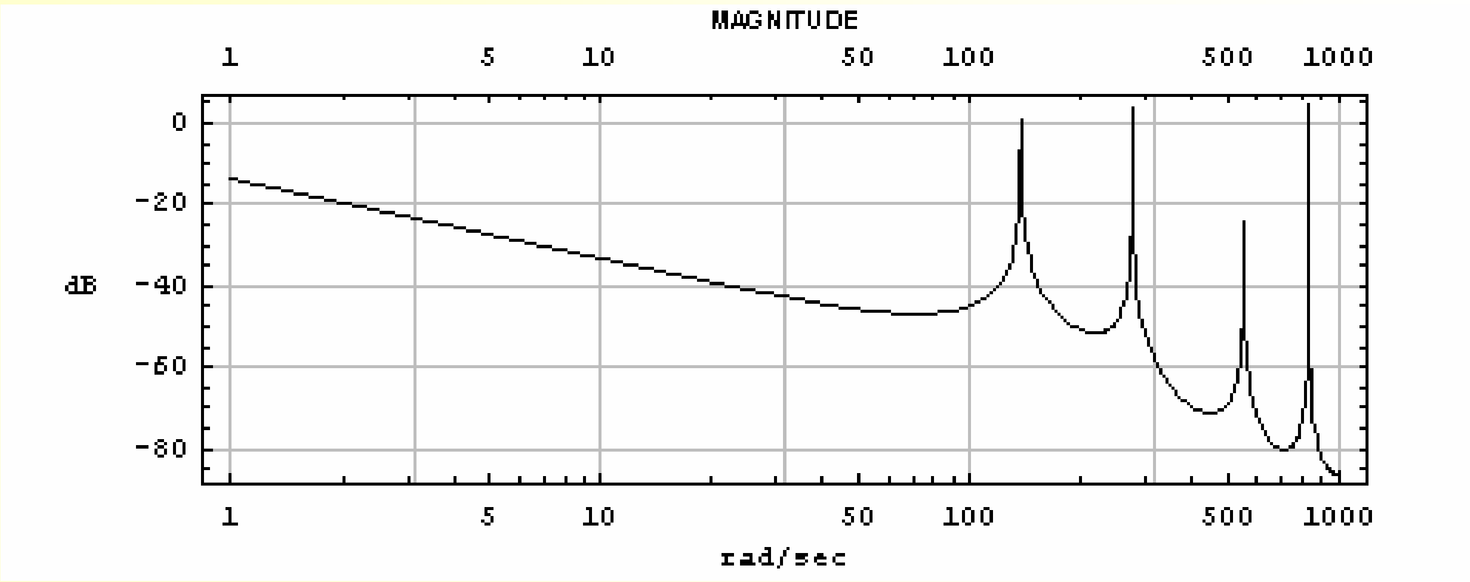


Suspension: Effect of Damping Ratio (velocity)



Suspension: road disturbance





c @ 50

8"

Road Disturbance Calculations

```
Y0[s_] :=
 Evaluate[LaplaceTransform[
   (8/12) UnitStep[v t] * Sin[v t] * UnitStep[π - v t], t, s]]
```

$$Y1[s\_] := \text{Evaluate}\left[\sum_{k=0}^{\infty} e\wedge(-k\,T\,s)\right]$$

```
Y2[s_] := Y0[s] Y1[s]
Bode[Y2, s, 0.1, 100]
```

Summary

- Time Response
  - Response to unit step input
  - Time response parameters, % overshoot, settling time
  - Pole locations
- Frequency Response
  - Response to sinusoidal input
  - Frequency transfer function
  - Bode plot
- Examples
  - Accelerometer
  - Vehicle suspension

Drexel UNIVERSITY