MEM 255 Introduction to Control Systems: *Review*

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What you should know

- **Computing Laplace transform pairs**
- **Computing time responses using partial fraction expansion**
- **Solving ode's using Laplace transforms**
- **Assembling state variable models**
- **Computing transfer functions from state space models**
- **Computing state space models from transfer functions**

The following are typical exam questions. Some with solutions.

Laplace Transform Pairs

Using the Laplace transform definition, derive the transform of the following time functions:

$$u(t), tu(t), \sin \omega t\, u(t), \cos \omega t\, u(t)$$

Using Laplace transform short table and theorems derive the transform of

$$t^3 u(t), e^{-at} \sin \omega t\, u(t), e^{-at} \cos \omega t\, u(t)$$

Example: Laplace Transform

$$\mathscr{L}\left[f(t)\right] = F(s) \triangleq \int_0^\infty f(t)e^{-st}dt$$

step: $\int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = 0 - \frac{1}{-s} = \frac{1}{s}$

ramp: $\int_0^\infty f(t)e^{-st}dt = \int_0^\infty te^{-st}dt$

recall 'integrtation by parts' $\int u dv = uv - \int v du$

$$\int_0^\infty te^{-st}dt = \int_0^\infty \frac{t}{-s} d\left(e^{-st}\right) = \left.\frac{t}{-s}e^{-st}\right|_0^\infty - \int_0^\infty e^{-st}\frac{dt}{-s} = (0-0) - \frac{1}{-s}\int_0^\infty e^{-st}dt = \frac{1}{s^2}$$

Example: Laplace Transform

$$\sin\omega t: \quad \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \sin\omega t e^{-st}dt$$

$$= \int_0^\infty -\frac{1}{\omega}e^{-st}d(\cos\omega t) = -\frac{1}{\omega}\left[e^{-st}\cos\omega t - \int_0^\infty \cos\omega t\left(-se^{-st}\right)dt\right]$$

$$= \frac{1}{\omega} - \frac{s}{\omega}\int_0^\infty \cos\omega t\, e^{-st}dt$$

$$= \frac{1}{\omega} - \frac{s}{\omega^2}\int_0^\infty e^{-st}d(\sin\omega t) = \frac{1}{\omega} - \frac{s}{\omega^2}\left[e^{-st}\sin\omega t\Big|_0^\infty + s\int_0^\infty \sin\omega t\, e^{-st}dt\right]$$

$$\int_0^\infty \sin\omega t e^{-st}dt = \frac{1}{\omega} - \frac{s^2}{\omega^2}\int_0^\infty \sin\omega t\, e^{-st}dt$$

$$\int_0^\infty \sin\omega t e^{-st}dt = \frac{\omega}{s^2+\omega^2}$$

Example: Laplace Transform

$e^{-at} \sin \omega t$:

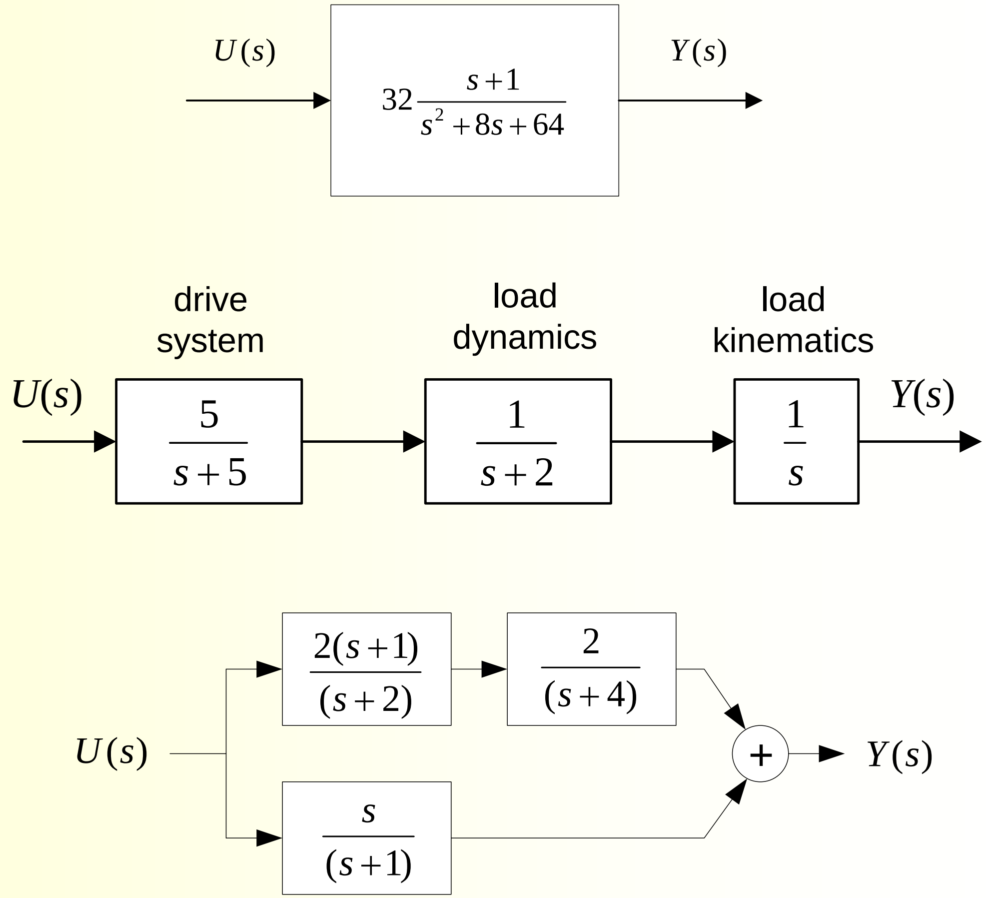
- from the short table: $\mathscr{L}\left[\sin \omega t\right] = \dfrac{\omega}{s^2 + \omega^2}$
- from Laplace transform Theorems: if

$$\mathscr{L}\left[f(t)\right] = F(s) \Rightarrow \mathscr{L}\left[e^{-at} f(t)\right] = F(s+a)$$

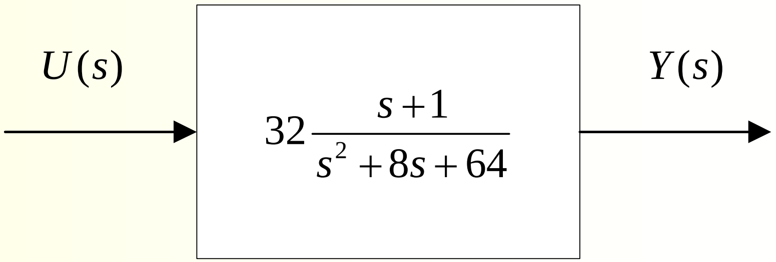
Thus, $\mathscr{L}\left[e^{-at} \sin \omega t\right] = \dfrac{\omega}{(s+a)^2 + \omega^2}$

Computing Time Response

Compute step and Impulse response of Systems on the right.



Example:



impulse response: $U(s)=1$

$$Y(s)=32\frac{s+1}{s^2+8s+64}=32\frac{s+1}{\left(s+4\left(1+j\sqrt{3}\right)\right)\left(s+4\left(1-j\sqrt{3}\right)\right)}$$

$$=\frac{c_1}{\left(s+4\left(1+j\sqrt{3}\right)\right)}+\frac{c_1^*}{\left(s+4\left(1-j\sqrt{3}\right)\right)}$$

$$y(t)=c_1e^{-4\left(1+j\sqrt{3}\right)t}+c_1^*e^{-4\left(1-j\sqrt{3}\right)t}=e^{-4t}\left(c_1e^{-j4\sqrt{3}t}+c_1^*e^{-j4\sqrt{3}t}\right)=e^{-4t}\left(2\,\mathrm{Re}\left(c_1e^{-j4\sqrt{3}t}\right)\right)$$

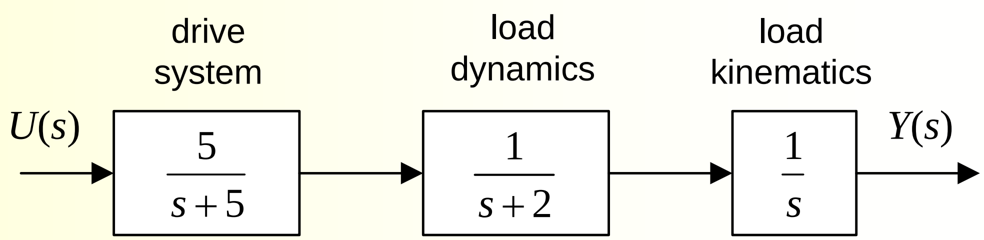
$$c_1=\lim_{s\to-4\left(1+j\sqrt{3}\right)}32\frac{\left(s+4\left(1+j\sqrt{3}\right)\right)(s+1)}{\left(s+4\left(1+j\sqrt{3}\right)\right)\left(s+4\left(1-j\sqrt{3}\right)\right)}=32\frac{\left(-4\left(1+j\sqrt{3}\right)+1\right)}{\left(-4\left(1+j\sqrt{3}\right)+4\left(1-j\sqrt{3}\right)\right)}$$

$$=32\frac{\left(-3-j4\sqrt{3}\right)}{-j8\sqrt{3}}=4\left(4-j\sqrt{3}\right)$$

$$c_1e^{-j4\sqrt{3}t}=4\left(4-j\sqrt{3}\right)\left(\cos 4\sqrt{3}t-j\sin 4\sqrt{3}t\right)=\left(16\cos 4\sqrt{3}t-4\sqrt{3}\sin 4\sqrt{3}t\right)+j(\cdots)$$

$$y(t)=2\left(16\cos 4\sqrt{3}t-4\sqrt{3}\sin 4\sqrt{3}t\right)$$

Example:



Step response: $U(s) = 1/s$

$$Y(s) = \frac{5}{(s+5)(s+2)s}U(s) = \frac{5}{(s+5)(s+2)s^2}$$

$$Y(s) = \frac{c_1}{s+5} + \frac{c_2}{s+2} + \frac{c_{31}}{s} + \frac{c_{32}}{s^2}$$

$$c_1 = \left.\frac{5}{(s+2)s^2}\right|_{s\to -5} = \frac{5}{7\cdot 25} = \frac{1}{35}, \quad c_2 = \left.\frac{5}{(s+5)s^2}\right|_{s\to -2} = \frac{5}{28}$$

$$c_{32} = \left.\frac{5}{(s+5)(s+2)}\right|_{s\to 0} = \frac{1}{2}, \quad c_{31} = \left.\frac{d}{ds}\left(\frac{5}{(s+5)(s+2)}\right)\right|_{s\to 0} = \left.\frac{5(7+2s)}{\left(10+7s+s^2\right)^2}\right|_{s\to 0} = \frac{35}{100}$$

$$y(t) = \frac{1}{35}e^{-5t}u(t) + \frac{5}{28}e^{-2t}u(t) + \frac{1}{2}u(t) + \frac{35}{100}tu(t)$$

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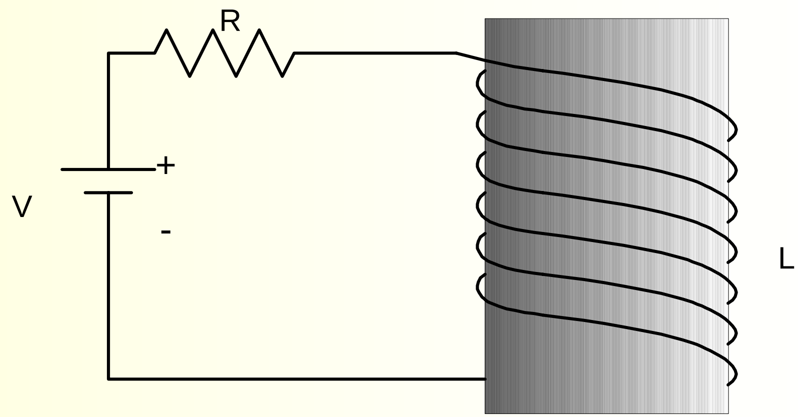
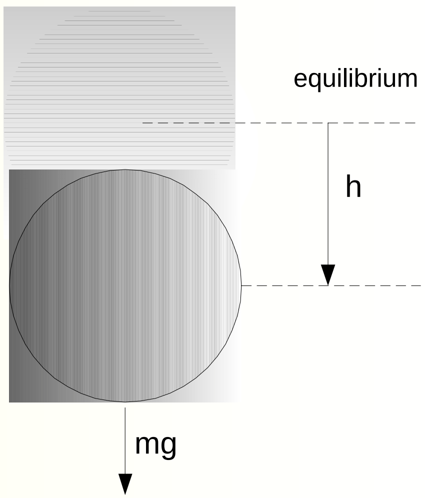
State Space Models

Determine the state space model for the magnetic levitation system shown. The linearized governing equations are:

$$m\frac{d^2h}{dt^2} = -\alpha i$$

$$V(t) = L\frac{di}{dt} + Ri$$





$$\frac{dh}{dt} = v$$

$$\frac{dv}{dt} = -\frac{\alpha}{m}i$$

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}V(t)$$

$$\Rightarrow \quad \frac{d}{dt}\begin{bmatrix} h \\ v \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\alpha/m \\ 0 & 0 & -R/L \end{bmatrix}\begin{bmatrix} h \\ v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} V(t)$$

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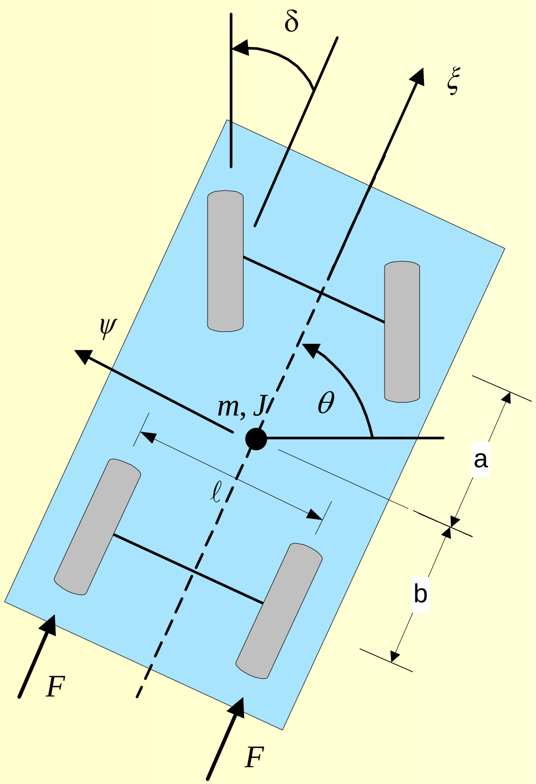
State Space Models



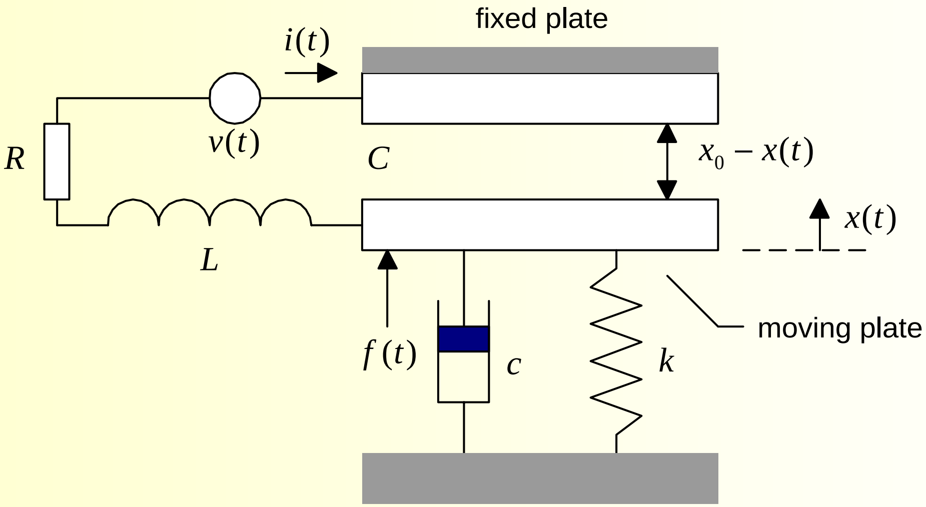
The linearized governing equations for the vehicle are:

$$m\ddot{y} = -2C_r\left(\frac{\dot{y}-b\dot{\theta}}{V_0}-\theta\right)-2C_f\left(\frac{\dot{y}+a\dot{\theta}}{V_0}-\theta-\delta\right)$$

$$J\ddot{\theta} = 2bC_r\left(\frac{\dot{y}-b\dot{\theta}}{V_0}-\theta\right)-2aC_f\left(\frac{\dot{y}+a\dot{\theta}}{V_0}-\theta-\delta\right)$$

Put the equations in state space form.

State Space Models



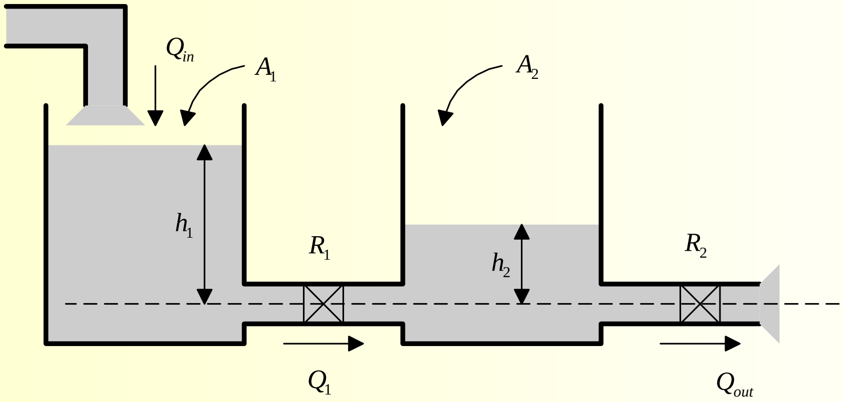
The governing equations for the capacitive microphone are

$$m\ddot{x} + c\dot{x} + kx - \frac{1}{\varepsilon A}(q - q_0) = f(t)$$

$$L\ddot{q} + R\dot{q} + \frac{1}{\varepsilon A}(q - q_0) = v(t)$$

Put them in state space form.

Transfer Function Models



Derive the transfer function $Q_{in} \rightarrow Q_{out}$ for the two tank system shown.

$$A_1 \frac{dh_1}{dt} = -\frac{1}{R_1}(h_1 - h_2) + Q_{in}$$

$$A_2 \frac{dh_2}{dt} = \frac{1}{R_1}(h_1 - h_2) - \frac{h_2}{R_2}$$

$$Q_{out} = \frac{h_2}{R_2}$$

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Example: Transfer Function

assume zero initial conditions

$$A_1 s H_1 = -\frac{1}{R_1}(H_1 - H_2) + Q_{in}, \quad A_2 s H_2 = \frac{1}{R_1}(H_1 - H_2) - \frac{1}{R_2}H_2$$

solve for H_2

$$\left(A_1 s + \frac{1}{R_1}\right)H_1 + \frac{1}{R_1}H_2 = Q_{in}$$

$$-\frac{1}{R_1}H_1 + \left(A_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right)H_2 = 0$$

$$\Rightarrow \quad R_1\left(A_1 s + \frac{1}{R_1}\right)\left(A_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right)H_2 + \frac{1}{R_1}H_2 = Q_{in}$$

$$H_2 = \frac{R_1}{R_1^2\left(A_1 s + \frac{1}{R_1}\right)\left(A_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) + 1}Q_{in}$$

$$Q_{out} = \frac{1}{R_2}H_2 = \frac{R_1 / R_2}{R_1^2\left(A_1 s + \frac{1}{R_1}\right)\left(A_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right) + 1}Q_{in}$$

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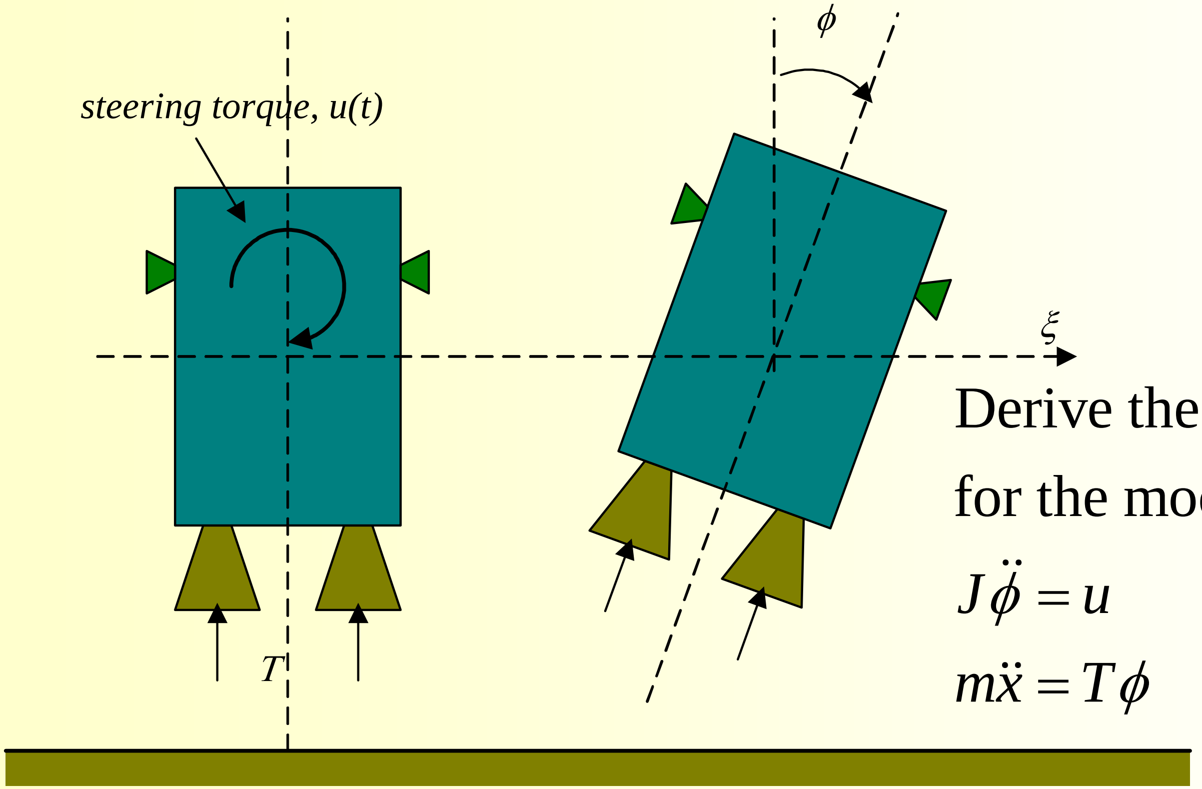
Transfer Function Models

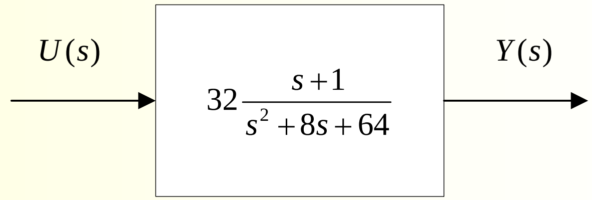


Derive the transfer function $u \rightarrow x$ for the moon lander

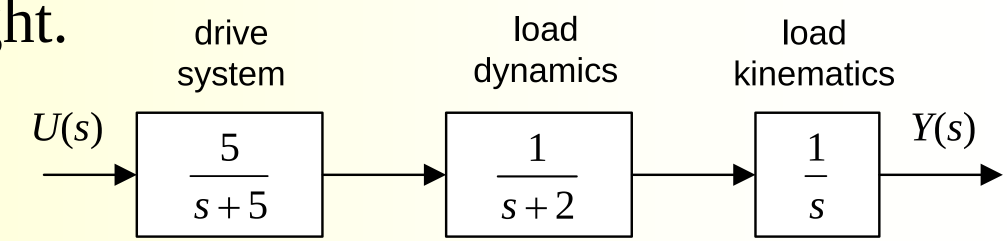
$$J\ddot{\phi} = u$$

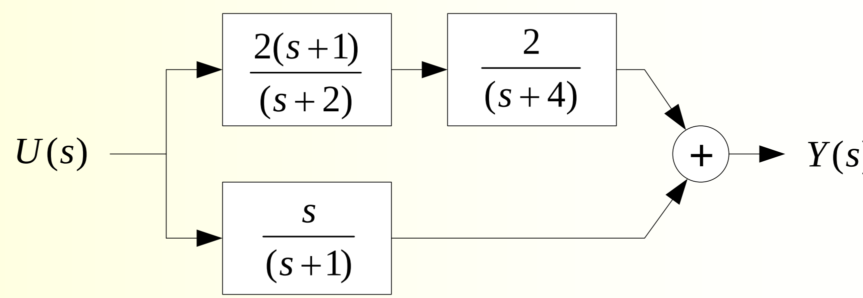
$$m\ddot{x} = T\phi$$

Transfer Function to State Space

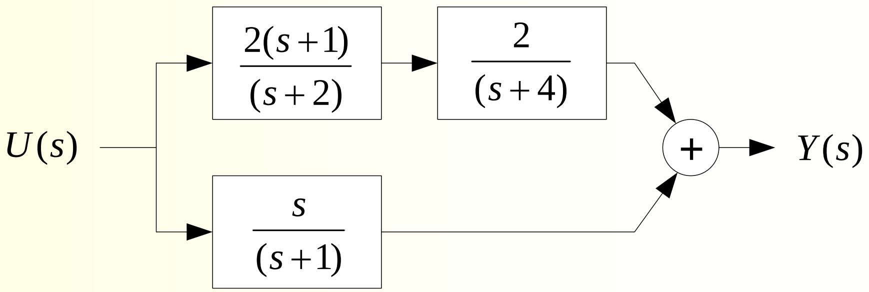


Derive state space models
For the systems on the right.





Example: tf 2 ss



$$Y(s) = \frac{s^3 + 10s^2 + 16s + 4}{s^3 + 7s^2 + 14s + 8} U(s)$$

$$Z(s) = \frac{1}{s^3 + 7s^2 + 14s + 8} U(s), Y(s) = \left(s^3 + 10s^2 + 16s + 4\right) Z(s)$$

$$\dddot{z} + 7\ddot{z} + 14\dot{z} + 8z = u, \; y = \dddot{z} + 10\ddot{z} + 16\dot{z} + 4z$$

$$x_1 = z$$

$$x_2 = \dot{x}_1 = \dot{z}$$

$$x_3 = \dot{x}_2 = \ddot{z}$$

$$\dot{x}_3 = -7x_3 - 14x_2 - 8x_1 + u, \; y = \left(-7x_3 - 14x_2 - 8x_1 + u\right) + 10x_3 + 16x_2 + 4x_1$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -7x_3 - 14x_2 - 8x_1 + u$$
$$y = -4x_1 + 2x_2 + 3x_3 + u$$

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Summary

- Computing Laplace transforms
- Inverting Laplace transforms via partial fraction expansion
- Computing system responses
- Building state variable models from governing equations
- Building transfer function models from governing equations
- Converting transfer functions to state variable models