MEM 255 Introduction to Control Systems: *Transfer function to state space*

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Outline

- **Problem definition: transfer function to state space**
- **Companion form**
  - **Two methods**
  - **Example**
    - **Hand calculation**
    - **MATLAB**
- **Diagonal Form**

Problem Definition

Given a transfer function:

$$G(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0},\ m\le n$$

Find a set of matrices A, B, C, D such that

$$G(s)=C\left[sI-A\right]^{-1}B+D$$

The answer, as we will, see is not unique. We will consider several approaches.

Note, we can always assume $m=n,$ then allow $b_n=0,\ldots,b_{m+1}=0.$

Companion Form

Consider the input-output relation:

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

$$\left(s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right) Y(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0\right) U(s)$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

We need to replace this n^{th}-order differential equation by a system of first order equations in the form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

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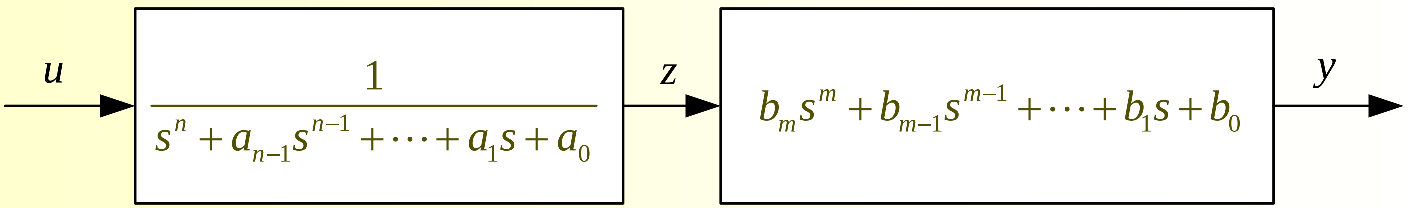
Method A, 1



Consider the auxiliary system:

$$z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_1\dot{z} + a_0 z = u$$

The output y can be obtained from

$$y = b_m z^{(m)} + b_{m-1}z^{(m-1)} + \cdots + b_1\dot{z} + b_0 z$$

Now, introduce the definitions

$$x_1 = z$$

$$x_2 = \dot{x}_1 = \dot{z}$$

$$\vdots$$

$$x_n = \dot{x}_{n-1} = z^{(n-1)}$$

Method A, 2

The auxilliary equation reduces to first order, i.e.,

$$\dot{x}_n + a_{n-1}x_n + \cdots + a_1 x_2 + a_0 x_1 = u$$

Also, the output can be written in terms of the states

$$\begin{aligned} y &= b_n z^{(n)} + b_{n-1} z^{(n-1)} + \cdots + b_1 \dot{z} + b_0 z \\ &= b_n \left\{ u - \left(a_{n-1}x_n + \cdots + a_1 x_2 + a_0 x_1 \right) \right\} + b_{n-1}x_n + \cdots b_1 x_2 + b_0 x_1 \\ &= \left(b_0 - b_n a_0 \right) x_1 + \cdots + \left(b_{n-1} - b_n a_{n-1} \right) x_n + b_n u \end{aligned}$$

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Method A, 3

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad b=\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C=\begin{bmatrix} (b_0-b_n a_0) & \cdots & \cdots & \cdots & (b_{n-1}-b_n a_{n-1}) \end{bmatrix}, \quad D=\begin{bmatrix} b_n \end{bmatrix}$$

Method B, 1

First, define n state variables, $x_1, x_2, \ldots, x_n$

$$x_1 = y - \alpha_1 u$$

$$x_2 = \dot{x}_1 - \alpha_2 u$$

$$\vdots$$

$$x_n = \dot{x}_{n-1} - \alpha_n u$$

Sequentially, use these definitions to eliminate $y, \dot{x}_1, \ldots, \dot{x}_n$

Method B, 2

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

$$\Downarrow$$

$$\left(x_1^{(n)} + \alpha_1 u^{(n)}\right) + a_{n-1}\left(x_1^{(n-1)} + \alpha_1 u^{(n-1)}\right) + \cdots + a_1\left(\dot{x}_1 + \alpha_1\dot{u}\right) + a_0\left(x_1 + \alpha_1 u\right)$$

$$= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

$$\Downarrow$$

$$\left(x_2^{(n-1)} + \alpha_2 u^{(n-1)} + \alpha_1 u^{(n)}\right) + a_{n-1}\left(x_2^{(n-2)} + \alpha_2 u^{(n-2)} + \alpha_1 u^{(n-1)}\right) + \cdots + a_1\left(\dot{x}_1 + \alpha_1\dot{u}\right) + a_0\left(x_1 + \alpha_1 u\right)$$

$$= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

$$\Downarrow \quad \Downarrow$$

$$\left(\dot{x}_n + \alpha_n\dot{u} + \alpha_{n-1}\ddot{u}\cdots + \alpha_1 u^{(n)}\right) + \cdots + a_1\left(x_2 + \alpha_2 u + \alpha_1\dot{u}\right) + a_0\left(x_1 + \alpha_1 u\right)$$

$$= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

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Method B, 3

Now, choose α_i's to eliminate u-derivatives $\dot{u}, \ddot{u}, \ldots, u^{(n)}$

$$\left(x_1^{(n)} + \alpha_1 u^{(n)}\right) + a_{n-1}\left(x_1^{(n-1)} + \alpha_1 u^{(n-1)}\right) + \cdots + a_1\left(\dot{x}_1 + \alpha_1 \dot{u}\right) + a_0\left(x_1 + \alpha_1 u\right)$$

$$= b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

$$\Downarrow$$

$$\dot{x}_n + a_{n-1} x_n + \cdots + a_1 x_2 + a_0 x_1$$

$$+ \alpha_1 u^{(n)}$$

$$+ \left(\alpha_2 + a_{n-1}\alpha_1\right) u^{(n-1)}$$

$$\vdots$$

$$+ \left(\alpha_n + a_{n-1}\alpha_{n-1} + \cdots + a_1\alpha_1\right)\dot{u}$$

$$+ \left(a_{n-1}\alpha_n + a_{n-2}\alpha_{n-1} + \cdots + a_0\alpha_1\right) u = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

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Method B, 4

$$\alpha_1 = b_n$$

$$\left(\alpha_2 + a_{n-1}\alpha_1\right) = b_{n-1}$$

$$\vdots$$

$$\left(\alpha_n + a_{n-1}\alpha_{n-1} + \cdots + a_1\alpha_1\right) = b_1$$

solve recursively for α's, then

$$\dot{x}_n + a_{n-1}x_n + \cdots + a_1x_2 + a_0x_1 = b_0u$$

Method B, 5

$$\dot{x}_1 = x_2 + \alpha_2 u$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + \alpha_n u$$
$$\dot{x}_n = -a_{n-1}x_n - \cdots - a_1 x_2 - a_0 x_1 + b_0 u$$

$$y = x_1 + \alpha_1 u$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \alpha_1 \end{bmatrix}$$

Example

$$G(s) = \frac{4s^2 + 5}{s^3 + 3s^2 + 2}$$

$$\dddot{y} + 3\ddot{y} + 2y = 4\ddot{u} + 5u$$

$$x_1 = y - \alpha_1 u,\ x_2 = \dot{x}_1 - \alpha_2 u,\ x_3 = \dot{x}_2 - \alpha_3 u$$

$$\Downarrow$$

$$\left(\dot{x}_3 + \alpha_3 \dot{u} + \alpha_2 \ddot{u} + \alpha_1 \dddot{u}\right) + 3\left(x_3 + \alpha_3 u + \alpha_2 \dot{u} + \alpha_1 \ddot{u}\right) + 2\left(x_1 + \alpha_1 u\right) = 4\ddot{u} + 5u$$

$\dddot{u}$ terms $\quad \alpha_1 = 0$

$\ddot{u}$ terms $\quad \alpha_2 = 4$

$\dot{u}$ terms $\quad \alpha_3 + 3\alpha_2 = 0 \Rightarrow \alpha_3 = -12$

$$\dot{x}_3 + 3\left(x_3 - 12u\right) + 2x_1 = 5u$$

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Example, Cont'd

$$\dot{x}_1 = x_2 + 4u$$

$$\dot{x}_2 = x_3 - 12u$$

$$\dot{x}_3 = -2x_1 - 3x_3 + 41u$$

$$y = x_1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -12 \\ 41 \end{bmatrix},$$

$$c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad d = 0$$

Example, MATLAB

```
>>s=tf('s');
>> G=(4*s^2+5)/(s^3+3*s^2+2);
>> ss(G)
```

```
a =
              x1      x2      x3
   x1         -3       0    -0.5
   x2          2       0       0
   x3          0       2       0
b =
          u1
   x1      2
   x2      0
   x3      0
c =
               x1        x2        x3
   y1           2         0     0.625
d =
          u1
   y1      0
```

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Diagonal Form

Consider the transfer function

$$G(s)=\frac{b_n s^n + b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

Assume distinct poles $\lambda_1,\ldots,\lambda_n$ and write partial fraction expandion

$$G(s)=c_0+\frac{c_1}{s+\lambda_1}+\cdots+\frac{c_n}{s+\lambda_n}$$

$$Y(s)=c_0U(s)+\frac{c_1}{s+\lambda_1}U(s)+\cdots+\frac{c_n}{s+\lambda_n}U(s)$$

Define $X_1(s),\ldots,X_n(s)$

$$X_1(s)=\frac{1}{s+\lambda_1}U(s),\ldots,X_n(s)=\frac{1}{s+\lambda_n}U(s)$$

Then

$$Y(s)=c_0U(s)+c_1X_1(s)+\cdots c_nX_2(s)$$

Diagonal Form, 2

From the definitions of $X_1, \ldots, X_2$

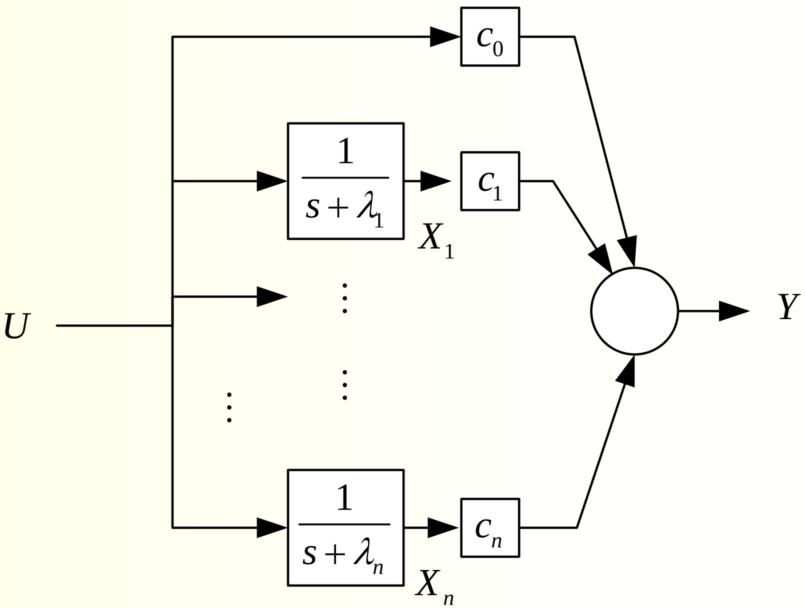
$$\dot{x}_1 = -\lambda_1 x_1 + u$$

$$\vdots \qquad\qquad y = c_1 x_1 + \cdots + c_n x_n + c_0 u$$

$$\dot{x}_n = -\lambda_n x_n + u$$

$$A = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & \lambda_3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \quad d = c_0$$



Example

$$G(s) = \frac{4s^2 + 5}{s^3 + 3s^2 + 2}$$

$$= c_0 + \frac{c_1}{s + 3.19582} + \frac{c_2}{s - 0.0979117 - 0.785003i} + \frac{c_3}{s - 0.0979117 + 0.785003i}$$

$$c_0 = 0$$

$$c_1 = 3.99943$$

$$c_2 = 0.00028498 - 0.497715i$$

$$c_3 = c_2^* = 0.00028498 + 0.497715i$$

Summary

- Transfer function to state space
- Companion form, 2 versions
  - Both give real coefficients
  - No need to factor numerator or denominator
- Diagonal form
  - Coefficients can be complex
  - Need to factor denominator