Assorted Notes

for

MEM 255: ***Introduction To Controls***

&

MEM 355: ***Performance Enhancement of Dynamic Systems***

June 11, 2003

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Preliminaries

These notes began with an attempt to bring coherence to what must seem like a disconnected and abstract assortment of concepts and to students who confront them for the first time. Unfortunately, they seem to grow each time I teach these courses to the point where I worry that they might confuse rather than clarify.

Control Engineering & Mathematics

Control engineering is a discipline dealing with the design of devices, called control systems, that influence the performance of a system through manipulation of control devices on the basis of observations of system behavior. Mechanisms of this sort have been employed for centuries but today they are truly ubiquitous. Control systems are an essential part of chemical and manufacturing processes, communication systems, electric power plants and systems, ground vehicles, ships, aircraft and spacecraft, robots and manipulators, computers and so on.

During the last century engineering has been transformed from a craft into a science. Those interested in profiting from society's thirst for new technology have found it impossible to rely on time consuming trial and error to develop new products or resolve problems in existing ones. Modern technologies like automobiles, aircraft, telecommunications, and computers are too complex to thrive solely on vast compilations of empirical data and decades of experience. Some intellectual constructs that organize and explain essential facts and principles are required. So engineering, in general, has come to adopt the style and methods of the natural sciences.

At the core of this point of view is the distinction between two thought processes: the *physical*, and the *mathematical*. While engineers conceive of problems in the physical world and construct solutions intended for application in the physical world, the solution is almost always developed in the mathematical world. Today's engineers must be comfortable with translating between them. In the mathematical domain we work with *abstractions* of the physical. Abstraction is essential because most systems or devices involve so many irrelevant attributes that their complete characterization would only obscure practical solutions. On the other hand, abstraction can be dangerous because it is often easy to overlook important features and consequently to develop designs that fail to perform adequately in the physical world. Herein lies the challenge and the art of engineering.

Because of its inter-disciplinary nature and the breadth of its applications, control engineering is especially reliant on a scientific perspective. Unifying principles that bring together seemingly diverse subjects within a single inclusive concept are of great significance. Mathematics, which may be regarded as the ultimate unifying principle in science and technology, is very much at the heart of control engineering. In some circles control theory is considered to be a branch of applied mathematics. But while mathematics is a necessary part of control engineering there is much more to it. A control system design project begins (with a problem definition) and ends (with a solution implemented) in the physical world.

A Little History

It is almost certain that feedback controllers in primitive form existed many centuries ago. But the earliest to receive prominence in the written history is James Watt's fly-ball governor, a device that received a patent in the late 18th century. Governors, or speed regulators, were important in many systems of the late 18th and 19th centuries during which time several alternatives were developed to meet increasingly stringent performance requirements. A paper by the noted physicist James Clerk Maxwell, "On Governors," published in 1868 is considered to be the first paper dealing directly with the theory of automatic control[1]. At that time period, the Russian engineer Vyshnegradskii worked on similar control problems, publishing his results in 1876-77 in French and German[2].

[1] This paper and several others noted below can be found reprinted in the books [1] and [2].

[2] Vyshnegradskii's work is summarized in the text book [3] written by a brilliant, if demented, mathematician.

Subsequent papers by several authors through the mid-1930's dealt with turbine speed control and other applications such as ship steering and stabilization, electric power system load and frequency control, navigation and aircraft autopilots. In addition to the contributions these papers made to their respective application domains, they led to a formal definition of the basic feedback control problem called the servomechanism design problem.

The 1930's also saw the development of *frequency response* methods for dealing with stability issues in feedback systems. While these techniques were developed in the context of feedback amplifier design and not control systems per se, the methods of Nyquist and Bode have become basic tools of control systems analysis. The WWII years produced many new results. Radar, fire control, navigation and communications problems pushed control system technology to its limits and beyond. For the first time "optimal" control design problems were posed and solved (using frequency domain methods) [4]. The most prominent example being the formulation and solution of the single input – single output optimal control problem, now known as the Weiner-Hopf-Kolmogorov problem.

The 1950's saw dissemination of the war years' efforts and also new results, including Evan's *root locus* method, were published. For the first time multiple-input/multiple-output control problems were formulated. Frequency (transform) domain methods were by now well entrenched. The 1960's saw this state of affairs turned upside down when R. E. Kalman argued that *time domain* or *state space* methods were more appropriate for multivariable and nonlinear control. Kalman and Bucy [5] solved the multivariable version of the Weiner-Hopf-Kolmogorov optimal control problem - in the time domain. Indeed, state space and optimal control methods seemed tailor made for the 'race to the moon' that ended in 1969.

By the mid 1970's the state space tidal wave seemed to have run its course and the virtues of the transform (frequency domain) point of view vis-a-vis robustness to model uncertainty came, once again, into focus. Through the 1970's and 1980's the theory attempted to reconcile the transform and time domain perspectives [6].

Impact of the Digital Computer

Not only has the theory of control engineering evolved quite substantially over the last few decades - driven largely by a dramatically expanding domain of application - but the tools of the discipline have also changed radically. In fact, one could argue that it is the availability of new tools for analysis, design and, particularly, implementation that underlies the pervasive inclusion of feedback control in all manner of present day systems and devices. Digital computers only became widely available in the 1960's and workstations and personal computers really came of age in the 1980's. During the past decade microprocessors have become so powerful and inexpensive that they have opened the door for applications of control not conceived of in earlier years.

Many systems and products require feedback control in order to function. Examples include computer disk drives, robots, spacecraft and some aircraft. It would be impractical and often impossible to operate modern manufacturing systems or power plants efficiently and safely without automatic control. But even consumer products from washing machines to CD players to automobiles, require or benefit in terms of cost and performance when actuation, sensing, and control are integrated in their design. In automobiles, for example, control systems are increasingly used in engines for improved efficiency and emission control, in airbags, anti-lock brakes, skid and traction control and in suspension systems for improving both rideability and handling – not to mention cruise control and climate control. So pervasive is the design of mechanical systems integrated with sensing, actuation and control that the name "mechatronics" has been coined to identify this branch of engineering.

Course Objectives

This sequence of courses is intended to provide a comprehensive introduction to the concepts, methods and practice of linear control systems analysis and design.

The specific objectives of **MEM 255: Introduction to Control Systems** are:

- Introduce time domain (state space) and transform domain (transfer function) models of linear dynamical systems.

- Develop the general process of deriving state pace models from physical principles.
- Introduce the methods of deriving transfer functions from state space models and vice versa.
- Introduce the basics of transform domain analysis: poles & zeros, the frequency transfer function, Bode Plots and working with block diagrams.
- Introduce the basics of time domain analysis: eigenvalues & eigenvectors, state transition matrix and the "variation of parameters" formula, modal analysis and similarity transformations.
- Develop concept of stability and tools for parametric stability analysis.
- Provide a comprehensive introduction to the control system computations using MATLAB.

The specific objectives of **MEM 355: Performance Enhancement of Dynamical Systems** are:

- Define the control system design problem and develop a preliminary appreciation of the tradeoffs involved and requirements for robust stability and performance.
- Develop concepts and tools for ultimate state error analysis.
- Develop the relationship between time domain and frequency domain performance specifications, e.g, rise time, overshoot, settling time, sensitivity function and bandwidth.
- Develop frequency domain design methods, including: the root locus method, Nyquist & Bode methods, and stability margins.
- Provide an introduction to state space design: controllability and obervability, pole placement, design via the separation principle (time permitting).
- Emphasize computational methods using MATLAB.

[1] G. J. Thaler, *Automatic Control: Classical Linear Theory*. Stroudsburg: Dowden, Hutchinson & Ross, 1974.

[2] R. Bellman and R. Kalaba, "Selected Papers on Mathematical Trends in Control Theory," . New York: Dover Publications, 1964.

[3] L. S. Pontryagin, *Ordinary Differential Equations*. Reading: Addison Wesley, 1962.

[4] N. Weiner, *The Extrapolation, Interpolation and Smoothing of Stationary Time Series*. New York: J. Wiley & Sons, 1949.

[5] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," *ASME Journal of Basic Engineering*, pp. 95-108, 1961.

[6] A. G. J. MacFarlane, *Frequency-Response methods in Control Systems*. New York: J. Wiley & Sons, 1979.

What is a Linear System?

A dynamical system is a process whose behavior evolves as a function of time, so that time needs to be considered as an independent variable. Typically, we associate a set of observed variables, or *outputs*, denoted $y(t)$, with the system. A dynamical system may be excited by time-varying stimuli, or *inputs*, denoted $u(t)$. A *mathematical model* of a dynamical system is a set of mathematical relations that describe how the inputs affect the outputs. Such models allow us to compute the output time function given the input time function. Suppose we observe a system operate over a time interval $[0,T]$. Then we can view the system as a mathematical function whose input is the function $u(t), t \in [0,T]$ and whose output is the function $y(t), t \in [0,T]$. Physical systems have the property that the output at a particular time t cannot depend on future inputs, i.e., on $u(\tau), \tau \in [t,T]$. This requirement, sometimes referred to as *causality*, imposes important restrictions on the mathematical model.

These courses concern *linear* systems (as opposed to more general *nonlinear* systems). Suppose we view a system as a mapping in the following way. Let $\mathbf{u}$, $\mathbf{y}$ denote the entire time functions over the interval $[0,T]$. Suppose we denote the mapping that models the system as

$$\mathbf{y} = F(\mathbf{u})$$

Consider two inputs and their corresponding outputs, $\mathbf{u}_1 \to \mathbf{y}_1$ and $\mathbf{u}_2 \to \mathbf{y}_2$. Let α, β be any two numbers and consider the input $\mathbf{u}_3 = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2$. The system is *linear* if and only if

$$\mathbf{y}_3 = F(\mathbf{u}_3) = F(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) = \alpha\mathbf{y}_1 + \beta\mathbf{y}_2$$

In words, the system (or its mathematical model) satisfies the *principle of superposition*. Notice that this definition of a linear system requires that

$$\mathbf{y} = F(k\mathbf{u}) = kF(\mathbf{u})$$

for any number *k*. Thus, the relation *F* is *homogenous*.

Laplace Transform Summary

Definition

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt\ , \qquad f(t) = L^{-1}[F(s)] = \tfrac{1}{2\pi j}\int_{-j\infty}^{j\infty} e^{st} F(s)ds$$

Very Short Table of Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
$e^{-\lambda t}u(t)$	$\dfrac{1}{s+\lambda}$
$\sin(\omega t)\,u(t)$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\,u(t)$	$\dfrac{s}{s^2+\omega^2}$

Basic Theorems

Linearity	$L[\alpha_1 f_1(t) + \alpha_2 f_2(t)] = \alpha_1 F_1(s) + \alpha_2 F_2(s)$
Time Shift	$L[f(t-T)] = e^{-sT}F(s)$
Frequency Shift	$L[e^{-at}f(t)] = F(s+a)$
Derivative	$L[\dot f(t)] = sF(s) - f(0)$
Final Value	$f(\infty) = \lim\limits_{s\to 0} sF(s)$
Initial Value	$f(0+) = \lim\limits_{s\to\infty} sF(s)$

Partial Fraction Expansion

The partial fraction expansion method breaks down strictly proper rational transfer functions into simple, easily inverted parts. Consider

$$Y(s) = k\frac{n(s)}{d(s)} = k\frac{\textit{monic poly of } \deg = m}{\textit{monic poly of } \deg = n > m}$$

$$n(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_0 = (s-\zeta_m)(s-\zeta_{m-1})\cdots(s-\zeta_1)$$

$$d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = (s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_1)$$

$$Y(s) = \frac{k\,n(s)}{(s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_1)}$$

Case 1: distinct roots, $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

When the denominator has n distinct roots, the transfer function can be expanded in the form with a unique set if coefficients, $c_i, i = 1,\ldots,n$. These constants are called **residues**.

$$Y(s) = \frac{k\,n(s)}{(s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_1)} = \frac{c_1}{s-\lambda_1} + \cdots + \frac{c_n}{s-\lambda_n}$$

To determine c_i

$$\frac{(s-\lambda_i)k\,n(s)}{(s-\lambda_n)\cdots(s-\lambda_i)\cdots(s-\lambda_1)} = \frac{(s-\lambda_i)c_1}{s-\lambda_1} + \cdots + \frac{(s-\lambda_i)c_i}{(s-\lambda_i)} + \cdots + \frac{(s-\lambda_i)c_n}{s-\lambda_n}$$

$$\frac{k\,n(s)}{(s-\lambda_n)\cdots 1\cdots(s-\lambda_1)} = \frac{(s-\lambda_i)c_1}{s-\lambda_1} + \cdots + c_i + \cdots + \frac{(s-\lambda_i)c_n}{s-\lambda_n}$$

Now set $s \to \lambda_i$, to obtain

$$c_i = \lim_{s\to\lambda_i}(s-\lambda_i)Y(s)$$

Case 2: Nondistinct roots.

Consider the special case, $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \cdots \neq \lambda_n$. In this case the transfer function can be expanded in the form

$$Y(s)=\frac{k\,n(s)}{(s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_3)(s-\lambda_1)^2}=\frac{c_{11}}{s-\lambda_1}+\frac{c_{12}}{(s-\lambda_1)^2}+\frac{c_3}{s-\lambda_3}+\cdots+\frac{c_n}{s-\lambda_n}$$

Each coefficient corresponding to a distinct root, i.e., $c_3,\ldots,c_n$ can be obtained as above. To determine c_{11},c_{12} we proceed as follows. Multiply by $(s-\lambda_1)^2$ to obtain

$$(s-\lambda_1)^2\,Y(s)=\frac{k\,n(s)}{(s-\lambda_n)(s-\lambda_{n-1})\cdots(s-\lambda_3)}=(s-\lambda_1)c_{11}+c_{12}\;+\frac{(s-\lambda_1)^2c_3}{s-\lambda_3}+\cdots+\frac{(s-\lambda_1)^2c_n}{s-\lambda_n}$$

so that

$$c_{12}=\lim_{s\to\lambda_1}(s-\lambda_1)^2\,Y(s)$$

$$c_{11}=\lim_{s\to\lambda_1}\frac{d}{ds}(s-\lambda_1)^2\,Y(s)$$

Similar calculations for a general term involving root λ_1 of order r leads to

$$Y(s)=\frac{n(s)}{(s-\lambda_1)^r}=\frac{c_{11}}{(s-\lambda_1)}+\frac{c_{12}}{(s-\lambda_1)^2}+\cdots+\frac{c_{1r}}{(s-\lambda_1)^r}$$

with

$$c_{1k}=\lim_{s\to\lambda_1}\frac{1}{(r-k)!}\frac{d^{r-k}}{ds^{r-k}}(s-\lambda_1)^r\,Y(s)$$

Example 1:

Consider the transform

$$Y(s)=\frac{5}{s^2+3s+2}=\frac{5}{(s+1)(s+2)}$$

We wish to compute $y(t)=L^{-1}[Y(s)]$.

$$Y(s)=\frac{c_1}{s+1}+\frac{c_2}{(s+2)}$$

$$c_1 = (s+1)Y(s)\big|_{s\to -1} = \left.\frac{5}{s+2}\right|_{s\to -1} = 5$$
$$c_2 = (s+2)Y(s)\big|_{s\to -1} = \left.\frac{5}{s+1}\right|_{s\to -2} = -5$$

$$Y(s) = \frac{5}{s+1} + \frac{-5}{(s+2)}$$

Now, we only need to use the transform pair: $1/(s+\lambda) \Leftrightarrow e^{-\lambda t}u(t)$

$$y(t) = L^{-1}[Y(s)] = \left(5e^{-t} - 5e^{-2t}\right)u(t)$$

Example 2:

Consider

$$Y(s) = \frac{2s+3}{s^3+2s^2+s} = \frac{2s+3}{s(s+1)^2}$$

We begin by writing

$$Y(s) = \frac{c_1}{s} + \frac{c_{21}}{s+1} + \frac{c_{22}}{(s+1)^2}$$

and compute

$$c_1 = sY(s)\big|_{s\to 0} = \left.\frac{2s+3}{(s+1)^2}\right|_{s\to 0} = 3$$
$$c_{22} = (s+1)^2\,Y(s)\big|_{s\to -1} = \left.\frac{2s+3}{s}\right|_{s\to -1} = -1$$

$$\begin{aligned} c_{11} &= \left.\frac{d}{ds}(s+1)^2\,Y(s)\right|_{s\to -1} \\ &= \left.\frac{d}{ds}\frac{2s+3}{s}\right|_{s\to -1} \\ &= \left.\left(2\frac{1}{s} - \frac{2s+3}{s^2}\right)\right|_{s\to -1} = -3 \end{aligned}$$

Thus,

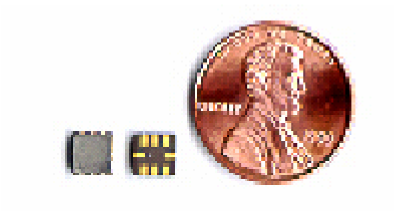
$$Y(s) = \frac{3}{s} + \frac{-3}{s+1} + \frac{-1}{(s+1)^2}$$

Now, $1/s \Rightarrow 1, 1/(s+1) \Rightarrow e^{-t}, 1/(s+1)^2 \Rightarrow te^{-t}$, so that

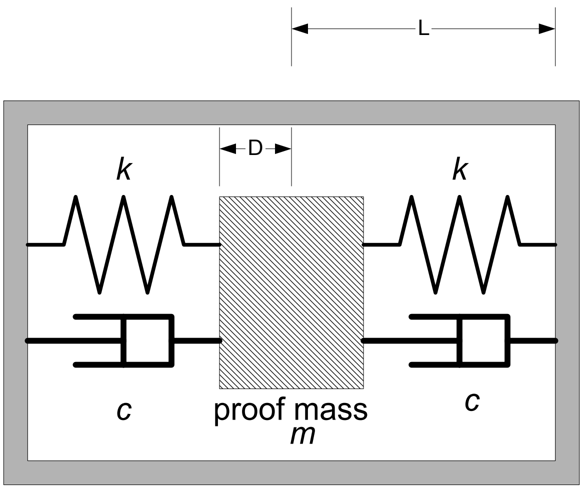
$$y(t) = 3 - 3e^{-t} - te^{-t}$$

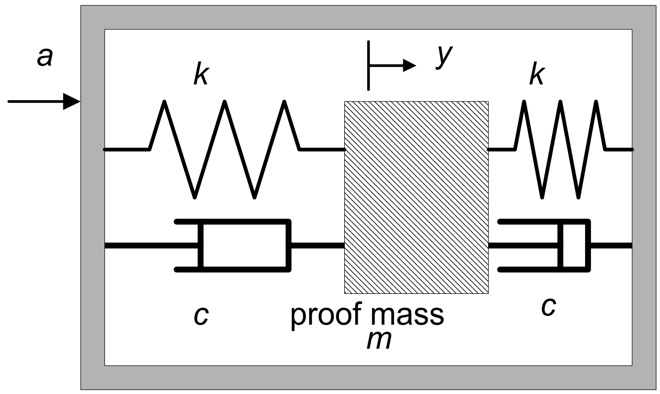
State Space & Transfer Function Models

Example: Accelerometer



Consider the system shown in the figure below





$$m\left(\ddot{y}+a\right)=\sum Forces=-2ky-2c\dot{y}$$

$$\Rightarrow m\ddot{y}=-2ky-2c\dot{y}-ma$$

Reduction to State Space Form

Define the new variable: $v=\dot{y}$

Rewrite the governing equation: $m\dot{v}=-2ky-2cv-ma$

$$\begin{aligned}\dot{y}&=v\\ \dot{v}&=-(2k/m)y-(2c/m)v-a\end{aligned}$$

rename the variables $y\rightarrow x_1, v\rightarrow x_2, a\rightarrow u$ to obtain

State Equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2k/m & -2c/m\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ -1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+[0]u$$

Notice that:

$$A=\begin{bmatrix}0 & 1\\ -2k/m & -2c/m\end{bmatrix},\ B=\begin{bmatrix}0\\ -1\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 0\end{bmatrix},\ D=[0]$$

The Transfer Function

Recall

$$L[\dot{y}] = sL[y] - y(0+) = sY(s) - y(0+)$$

$$L[\ddot{y}] = sL[\dot{y}] - \dot{y}(0+) = s^2 Y(s) - sy(0+) - \dot{y}(0+)$$

Assume $y(0+) = 0$ and $\dot{y}(0+) = 0$ in which case

$$m\ddot{y} = -2ky - 2c\dot{y} - ma \Rightarrow ms^2 Y(s) = -2kY(s) - 2csY(s) - mA(s)$$

or

$$Y(s) = -1 \frac{1}{s^2 + (2c/m)s + (2k/m)} A(s)$$

The transfer function is:

$$G(s) = -1 \frac{1}{s^2 + (2c/m)s + (2k/m)}$$

Analyzing The 2nd Order System

We will investigate the response of a second order system whose transfer function is

$$G(s) = \frac{\omega_n^2}{s^2 + 2\rho s + \omega_n^2}$$

By rescaling the output and defining new parameters, the accelerometer transfer function can be put in this form. First, reparameterize the accelerometer denominator

$$s^2 + (2c/m)s + (2k/m) = s^2 + 2\rho\omega_n s + \omega_n$$

where we define the *undamped natural frequency*, ω_n and the *damping ratio* ρ:

$$\omega_n = \sqrt{2k/m}\,, \quad \rho = \frac{c}{m}\sqrt{\frac{m}{2k}} = c\sqrt{\frac{1}{2km}}$$

Now, replace y by a scaled version $y_s(t) = -\omega_n^2 y(t)$. The transfer function $A(s) \to Y_s(s)$ is given by $G(s)$ defined above.

In fact, since we can rescale the independent variable s there is no harm in doing calculations with $\omega_n = 1$.

Time Domain: Response to a Step Input

Suppose the input is a unit step. Then the output is

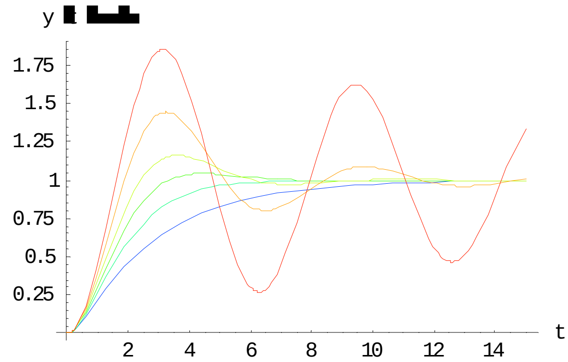
$$Y(s) = G(s) = \frac{1}{(s^2 + 2\rho s + 1)s}$$

and

$$y(t) = L^{-1}\left[G(s)\right] = \frac{-2\sqrt{-1+\rho^2} + \left(-\rho + \sqrt{-1+\rho^2}\right)e^{-t\left(\rho+\sqrt{-1+\rho^2}\right)} + \left(\rho + \sqrt{-1+\rho^2}\right)e^{-t\left(\rho-\sqrt{-1+\rho^2}\right)}}{2\sqrt{-1+\rho^2}}$$

$$y(t) = 1 - \frac{e^{-t\rho}}{\sqrt{1-\rho^2}}\left(\sqrt{1-\rho^2}\cos\left[t\sqrt{1-\rho^2}\right] + \rho\sin\left[t\sqrt{1-\rho^2}\right]\right)$$

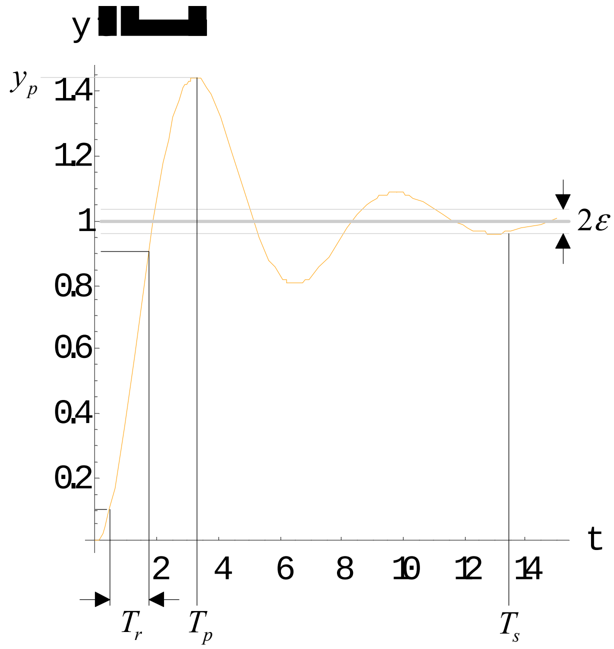
The following plot shows trajectories for various values of damping, $\rho = (1.5, 1.0, 0.707, 0.5, 0.25, 0.05)$



Several parameters are commonly used to characterize the shape of a step response trajectory. Sometimes performance specifications are stated in terms of these quantities. Definitions can vary, however. Here are a few basic quantities.

- *rise time*, T_r, usually defined as the time to get from 10% to 90% of its ultimate (i.e., final) value.
- *settling time*, T_s, the time at which the trajectory first enters an ε-tolerance of its ultimate value and remains there (ε is often taken as 2% of the ultimate value).
- *peak time*, T_p, the time at which the trajectory attains its peak value.
- *peak overshoot*, *OS*, the peak or supreme value of the trajectory ordinarily expressed as a percentage of the ultimate value of the trajectory. An overshoot of more than 30% is often considered undesirable. A system without overshoot is 'overdamped' and may be too slow (as measured by rise time and settling time).

These terms are often used to describe trajectories for systems of higher order. They are most useful when the higher order dynamics are dominated by 2nd order effects.



Settling Time Formula

Notice that

$$|y(t)-1| \le \frac{e^{-t\rho\omega_n}}{\sqrt{1-\rho^2}}$$

Therefore

$$\frac{e^{-T_s\rho\omega_n}}{\sqrt{1-\rho^2}} = \varepsilon \Rightarrow |y(t)-1| < \varepsilon \ \forall t > T_s$$

Consequently

$$T_s = \frac{-\ln\left(\varepsilon\sqrt{1-\rho^2}\right)}{\rho\omega_n}$$

Overshoot Formula

To find the extremal points set $\dot{y} = 0$ to find

$$-e^{-t\left(\rho+\sqrt{-1+\rho^2}\right)\omega_n} + e^{t\left(\rho+\sqrt{-1+\rho^2}\right)\omega_n} = 0$$

which yields (assuming the system is underdamped, i.e., $\rho < 1$)

$$t = 0, t = \frac{\pi}{\sqrt{1-\rho^2}\,\omega_n}$$

Clearly, the maximum value occurs at

$$T_p = \frac{\pi}{\sqrt{1-\rho^2}\,\omega_n}$$

Now, compute

$$y(T_p) = 1 + e^{-\frac{\pi\rho}{\sqrt{1-\rho^2}}}$$

so that

$$OS = \frac{y(T_p) - 1}{1} \times 100 = e^{-(\rho\pi/\sqrt{1-\rho^2})} \times 100$$

Frequency Domain: Response to Sinusoidal Inputs

In order to better appreciate how an accelerometer (or any other system) works, we will consider how it responds to a sinusoidal input. In other words let's assume that

$$a(t) = C \sin \omega t$$

and investigate the output response $y(t)$.

Trick: Instead of using the input $\sin \omega t$ use the input $e^{j\omega t}$. Recall that $e^{j\omega t} = \cos \omega t + \sin \omega t$. This is useful because (i) the response of the system to a real input must be real and (ii) the principle of superposition holds for linear systems. It follows that the response to this complex input will be complex but the real part of the response corresponds to the real part of the input and the imaginary part of the response corresponds to the imaginary part of the input.

$$e^{j\omega t} = \cos \omega t + j \sin \omega t \rightarrow y(t) = y_{real}(t) + j y_{imag}(t)$$

$\Rightarrow$ $\cos \omega t \rightarrow y_{real}(t)$, and $\sin \omega t \rightarrow y_{imag}(t)$ $\sin \omega t \rightarrow -y_{imag}(t)$

Now,

$$Y(s) = G(s)L[e^{j\omega t}] = \frac{1}{s^2 + 2\rho s + 1}\frac{1}{s - j\omega}$$

but

$$\frac{1}{s^2 + 2\rho s + 1}\frac{1}{s - j\omega} = \frac{c_1}{s^2 + 2\rho s + 1} + \frac{c_2}{s - j\omega}$$

The response can be interpreted in terms of these two terms as the "transient" response and the "steady-state" response:

$$Y(s) = \underbrace{\frac{c_1}{s^2 + 2\rho s + 1}}_{transient\ response} + \underbrace{\frac{c_2}{s - j\omega}}_{steady-state\ response} \quad \text{or} \quad y(t) = y_{trans}(t) + y_{ss}(t)$$

If the poles of the system (roots of $s^2 + 2\rho s + 1$) have negative real parts (it does if ρ>0), then $y_{trans}(t) \to 0$ as $t \to \infty$. Also

$$y_{ss}(t) = c_2 e^{j\omega t}$$

We can easily compute c_1, c_2, but for the steady-state response we only need c_2:

$$c_2 = \lim_{s \to j\omega}(s - j\omega)G(s)\frac{1}{s - j\omega} = G(j\omega)$$

With ω given as a real number, $G(j\omega)$ evaluates to a complex number which we can write in polar form.

$$G(j\omega) = |G(j\omega)|e^{j\angle G(j\omega)} \xrightarrow[write\ as]{} \rho(\omega)e^{j\phi(\omega)}$$

Then

$$y_{ss}(t) = \rho(\omega)e^{j(\omega t + \phi(\omega))} = \rho(\omega)\cos(\omega t + \phi(\omega)) + j\rho(\omega)\sin(\omega t + \phi(\omega))$$

In summary:

input: $\cos(\omega t)$ produces steady-state output $\rho(\omega)\cos(\omega t + \phi(\omega))$

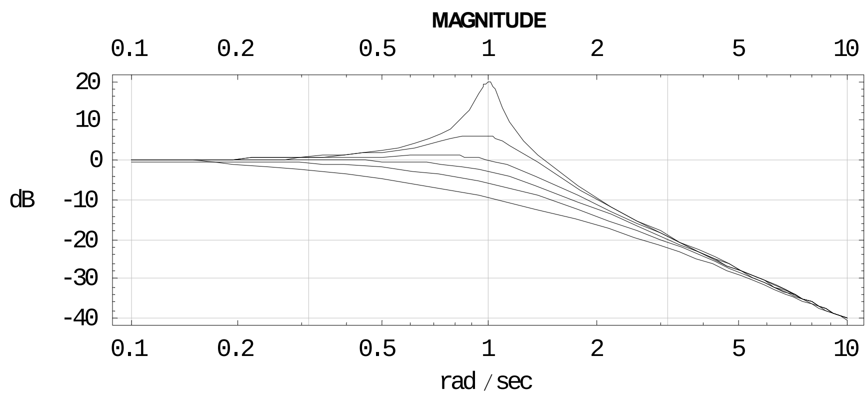
input: $\sin(\omega t)$ produces steady-state output $\rho(\omega)\sin(\omega t + \phi(\omega))$

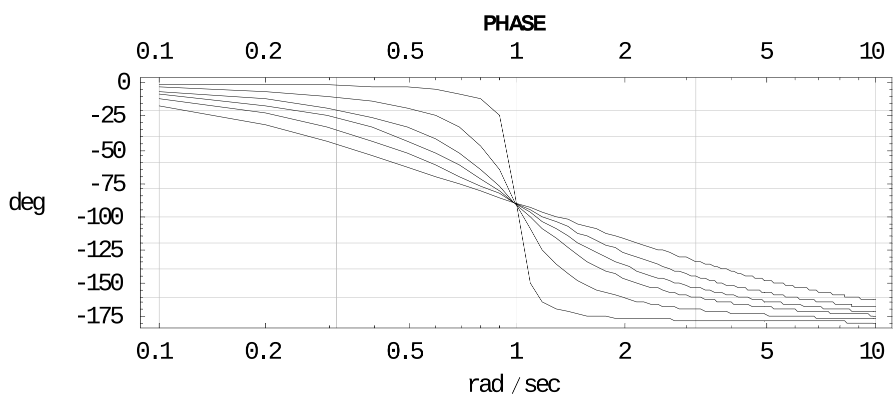
In other words the system produces a gain of $\rho(\omega)$ and a phase shift of $\phi(\omega)$ in the measurement of a sinusoidal acceleration, where $\rho(\omega)$ is the magnitude of the transfer function $G(s)$ evaluated at $s = j\omega$ and $\phi(\omega)$ is the phase of $G(s)$ evaluated at $s = j\omega$.

The function $G(j\omega)$ is called the *frequency transfer* function. Graphs that display the magnitude and phase angle of $G(j\omega)$ for $\omega > 0$ are called *Bode plots* of $G(s)$. The following are Bode plots of

$$G(s) = \frac{1}{(s^2 + 2\rho s + 1)}$$

that show the magnitude and phase for various values of damping, $\rho = (1.5, 1.0, 0.707, 0.5, 0.25, 0.05)$





Consider the frequency transfer function $G(j\omega)$. The *bandwidth* ω_{bw} of $G(j\omega)$ is the largest value of frequency such that $|G(j\omega)| > |G(0)| / \sqrt{2}$.

Consider the system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\rho s + \omega_n^2}$$

Compute

$$|G(j\omega)| = \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\rho^2\omega^2\omega_n^2}}$$

Set

$$\frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega_{bw}^2\right)^2 + 4\rho^2\omega_{bw}{}^2\omega_n^2}} = \frac{|G(0)|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Squaring and rearranging

$$2\omega_n^4 = \omega_n^4 + 2\left(-1 + 2\rho^2\right)\omega_n^2\omega_{bw}^2 + \omega_{bw}^4$$

Thus, solving for ω_{bw} we can show that the bandwidth is inversely related to the peak time and settling time

$$\begin{aligned}
\omega_{bw} &= \omega_n\sqrt{\left(1-2\rho^2\right) + \sqrt{4\rho^4 - 4\rho^2 + 2}} \\
&= \frac{-\ln\left(\varepsilon\sqrt{1-\rho^2}\right)}{T_s\rho}\sqrt{\left(1-2\rho^2\right) + \sqrt{4\rho^4 - 4\rho^2 + 2}} \\
&= \frac{\pi}{T_p\sqrt{1-\rho^2}}\sqrt{\left(1-2\rho^2\right) + \sqrt{4\rho^4 - 4\rho^2 + 2}}
\end{aligned}$$

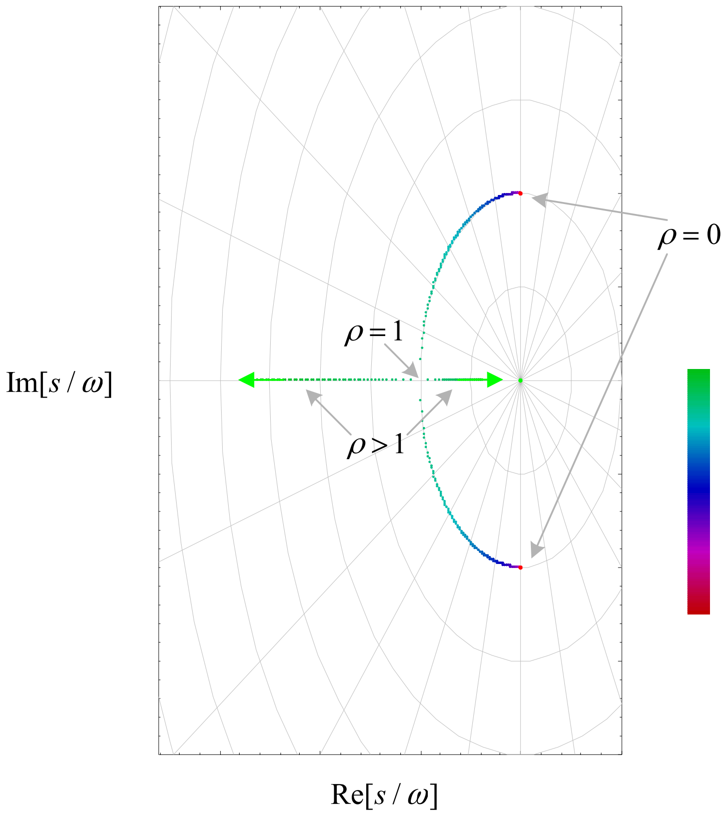
Factoring the Quadratic: $s^2 + 2\rho\omega_n s + \omega_n^2$

Now, apply the quadratic formula to $s^2 + 2\rho\omega_n s + \omega_n = 0$ to obtain

$$s_1 = \left(-\rho + \sqrt{-1+\rho^2}\right)\omega,\ s_2 = \left(-\rho - \sqrt{-1+\rho^2}\right)\omega$$

If $0 \le \rho < 1$, then the roots are imaginary. In this event it is convenient to express them in polar coordinates

$$s_{1,2} = \omega e^{\pm j\theta}, \theta = \sin^{-1}\rho, 0 \le \rho < 1$$


$\rho = 0$
$\rho = 1$
$\rho > 1$
Im[s/ω]
Re[s/ω]

Reduction of an n^th Order ODE to State Equations

Consider a system with output $y(t)$ and input $u(t)$. Suppose it is modeled by a single n^{th}-order ordinary differential equation of the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

with $m \le n$. Our goal is to replace this equation by a state variable model, that is a vector first order system of the form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

In the following paragraphs the general approach is described and an example is given. The particular definition of states used here are sometimes called "phase variables."

General Approach

Define *n* states $x_1, x_2, \ldots, x_n$ via

$$\begin{aligned} x_1 &= y - \alpha_1 u \\ x_2 &= \dot{x}_1 - \alpha_2 u \\ &\vdots \\ x_n &= \dot{x}_{n-1} - \alpha_n u \end{aligned}$$

The α_i's are constants to be specified later. Sequentially use these definitions to replace *y*, then derivatives of x_1, then derivatives of x_2, etc., up to derivatives of x_{n-1}:

$$(x_1^{(n)} + \alpha_1 u^{(n)}) + \cdots + a_1(\dot{x}_1 + \alpha_1\dot{u}) + a_0(x_1 + \alpha_1 u) = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u$$

$$(x_2^{(n-1)} + \alpha_2 u^{(n-1)} + \alpha_1 u^{(n)}) + \cdots + a_1(x_2 + \alpha_2 u + \alpha_1\dot{u}) + a_0(x_1 + \alpha_1 u) = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u$$

$$\vdots$$

$$(\dot{x}_n + \alpha_n\dot{u} + \alpha_2 u^{(n-1)} + \alpha_1 u^{(n)}) + \cdots + a_1(x_2 + \alpha_2 u + \alpha_1\dot{u}) + a_0(x_1 + \alpha_1 u) = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u$$

Now choose the α_i's to eliminate any derivatives of *u* that appear on the right hand side. Notice that lower indices of α knock out higher derivatives of *u*. On the left hand side the coefficients are obtained by collecting terms.

$$
\begin{aligned}
&\dot{x}_n + a_{n-1}x_n \cdots + a_1 x_2 + a_0 x_1 \\
&+ \alpha_1 u^{(n)} \\
&+ (\alpha_2 + a_{n-1}\alpha_1)u^{(n-1)} + \\
&\vdots \\
&+ (\alpha_n + a_{n-1}\alpha_{n-1} + \cdots + a_1\alpha_1)\dot{u} \\
&+ (a_{n-1}\alpha_n + a_{n-2}\alpha_{n-1} + \cdots + a_0\alpha_1)u = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u
\end{aligned}
$$

By starting with the $u^{(n)}$ term and working down to the $\dot{u}$ term we can sequentially compute the α_i's from $i = 1,\ldots,n$. The first nonzero α_i is $i = n - m + 1$.

Once the α_i's are obtained the state space model is:

$$
\begin{aligned}
&\dot{x}_1 = x_2 + \alpha_2 u \\
&\vdots \\
&\dot{x}_{n-1} = x_n + \alpha_n u \\
&\dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + \left(b_0 - (a_{n-1}\alpha_n + a_{n-2}\alpha_{n-1} + \cdots + a_0\alpha_1)\right)u \\
&y = x_1
\end{aligned}
$$

Notice that the first $n-1$ state equations and the output equation are the definitions of the states.

Example

Consider the system

$$\dddot{y} + 3\ddot{y} + 2y = 4\ddot{u} + 5u$$

Define the state variables

$$
\begin{aligned}
x_1 &= y - \alpha_1 u \\
x_2 &= \dot{x}_1 - \alpha_2 u \\
x_3 &= \dot{x}_2 - \alpha_3 u
\end{aligned}
$$

Substitute and obtain:

eliminate y with definition of x_1

$$(\dddot{x}_1 + \alpha_1\dddot{u}) + 3(\ddot{x}_1 + \alpha_1\ddot{u}) + 2(x_1 + \alpha_1 u) = 4\ddot{u} + 5u$$

eliminate $\dot{x}_1$ with definition of x_2 $\Downarrow$

$$(\ddot{x}_2 + \alpha_2 \ddot{u} + \alpha_1 \dddot{u}) + 3(\dot{x}_2 + \alpha_2 \dot{u} + \alpha_1 \ddot{u}) + 2(x_1 + \alpha_1 u) = 4\ddot{u} + 5u$$

eliminate $\dot{x}_2$ with definition of x_3 $\Downarrow$

$$(\dot{x}_3 + \alpha_3 \dot{u} + \alpha_2 \ddot{u} + \alpha_1 \dddot{u}) + 3(x_3 + \alpha_3 u + \alpha_2 \dot{u} + \alpha_1 \ddot{u}) + 2(x_1 + \alpha_1 u) = 4\ddot{u} + 5u$$

Now, choose α's to eliminate derivatives of u

$\dddot{u}$ terms $\alpha_1 = 0$

$\ddot{u}$ terms $\alpha_2 = 4$

$\dot{u}$ terms $\alpha_3 + 3\alpha_2 = 0 \Rightarrow \alpha_3 = -12$

Thus, the original equation reduces to

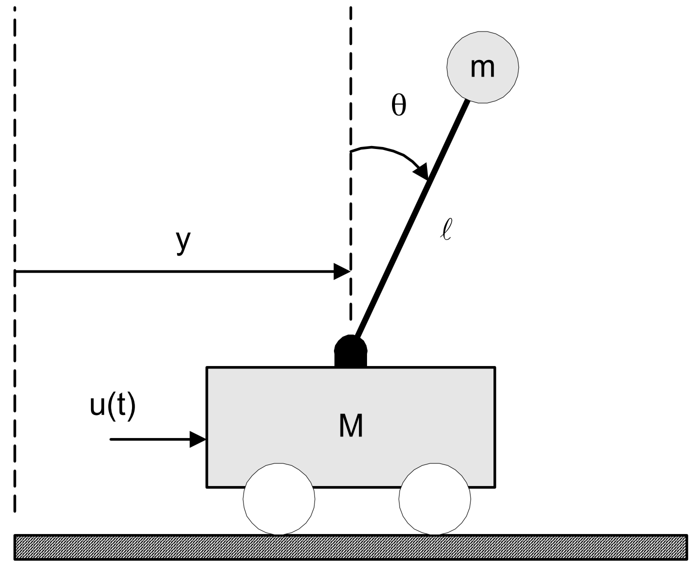
$$\dot{x}_3 + 3(x_3 - 12u) + 2x_1 = 5u$$

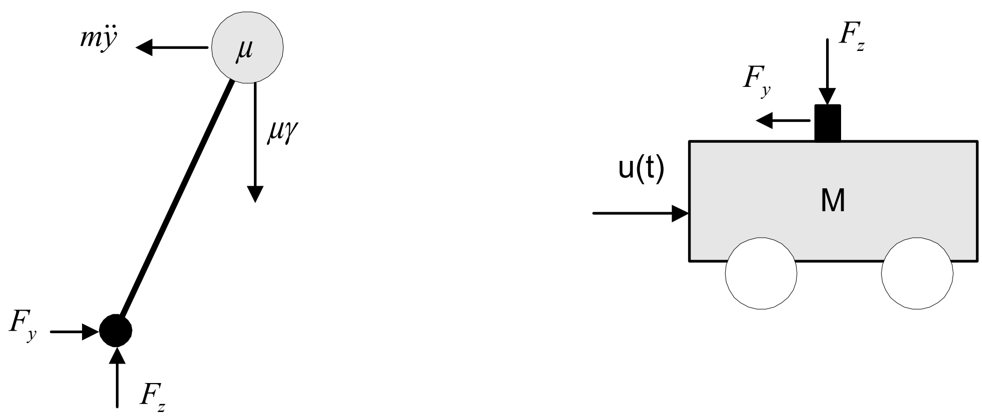
Combining this with the state definitions yields the state variable model:

$$\begin{aligned} \dot{x}_1 &= x_2 + 4u \\ \dot{x}_2 &= x_3 - 12u \\ \dot{x}_3 &= -2x_1 - 3x_3 + 41u \end{aligned}$$

$$y = x_1$$

Example: Inverted Pendulum





Equations via Newton's Law

Pendulum

Moments about joint: $\ell^2 m\ddot{\theta} = \ell mg\theta - \ell m\ddot{y}$

Forces in y-direction: $m\ell\ddot{\theta} + m\ddot{y} = F_y$

Carriage

Forces in y-direction: $M\ddot{y} = -F_y + u(t) \Rightarrow (M+m)\ddot{y} + m\ell\ddot{\theta} = u(t)$

$$\boxed{\begin{array}{c} \ddot{y} + \ell\ddot{\theta} - g\theta = 0 \\ (M+m)\ddot{y} + m\ell\ddot{\theta} = u(t) \end{array}}$$

Equations via Lagrange's Equations

Kinetic Energy: $\begin{aligned} T &= \tfrac{1}{2}\left[M\dot{y}^2 + m(\dot{y} + \ell\dot{\theta}\cos\theta)^2 + m(\ell\dot{\theta}\sin\theta)^2\right] \\ &\approx \tfrac{1}{2}\left[M\dot{y}^2 + m(\dot{y} + \ell\dot{\theta})^2\right] \end{aligned}$

Potential Energy: $\begin{aligned} V &= mg\left[\ell - \ell\cos\theta\right] \\ &\approx -\tfrac{1}{2} mg\ell\theta^2 \end{aligned}$

Virtual Work: $\delta W = u\delta y = \begin{bmatrix} 0 & u \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta y \end{bmatrix}$

Lagrangian: $L = T - V = \tfrac{1}{2}\left[M\dot{y}^2 + m(\dot{y} + \ell\dot{\theta})^2\right] + \tfrac{1}{2} mg\ell\theta^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) = 0 \Rightarrow m\ell(\ddot{y} + \ell\ddot{\theta}) - mg\ell\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right) - \left(\frac{\partial L}{\partial y}\right) = u(t) \Rightarrow (M+m)\ddot{y} + m\ell\ddot{\theta} = u(t)$$

$$\boxed{\begin{array}{c} \ddot{y} + \ell\ddot{\theta} - g\theta = 0 \\ (M+m)\ddot{y} + m\ell\ddot{\theta} = u(t) \end{array}}$$

Reduction to State Space

Define: $\begin{aligned} \frac{dy}{dt} &= v \\ \frac{d\theta}{dt} &= \omega \end{aligned}$ & rewrite $\begin{array}{c} \dot{v} + \ell\dot{\omega} = g\theta \\ (M+m)\dot{v} + m\ell\dot{\omega} = u(t) \end{array}$ $\Rightarrow$ $\begin{aligned} \dot{v} &= -\frac{mg}{M}\theta + \frac{1}{M}u \\ \dot{\omega} &= \frac{(M+m)g}{M\ell}\theta - \frac{1}{M\ell}u \end{aligned}$

$$\frac{d}{dt}\begin{bmatrix} y \\ \theta \\ v \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{M\ell} & 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ \theta \\ v \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{M\ell} \end{bmatrix} u$$

Control Design

Preliminary Examples

Cruise Control

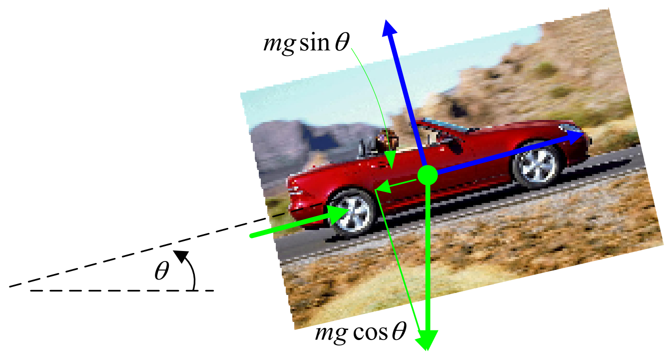


Figure 1. Force diagram for vehicle cruise control problem.

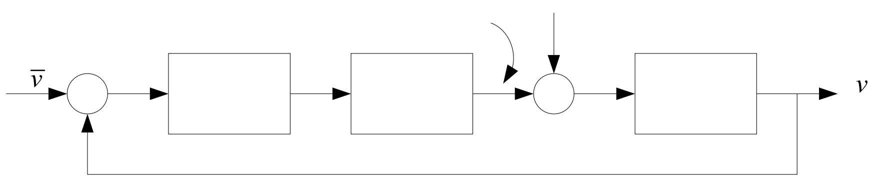


Figure 2. Block diagram for cruise control system.

vehicle: $m\dot{v} = F - mg\sin\theta(t) - cv \quad (g = 9.80665\ m/s^2)$

$$\dot{v} = \left(\frac{F_{\max}}{m}\right)\left(\frac{F}{F_{\max}}\right) - g\sin\theta - \left(\frac{c}{m}\right)v$$

Assume:

- θ small
- $(F_{\max}/m) = 1, \quad (c/m) = 0.02$

so that

$\dot{v} + 0.02v = u - 9.8\theta$

where

$v\,[m/s]$ speed (10 m/s=36 km/h=22 miles/hr)
u normalized throttle $0 \le u \le 1$
$\theta\,[rad]$ roadway slope

Choose a "proportional" + "integral control"

$$u(t) = k_p\left(\bar{v}(t) - v(t)\right) + k_i \int_0^t \left(\bar{v}(\tau) - v(\tau)\right) d\tau$$

The closed loop: Let $e(t) = \bar{v}(t) - v(t)$,

$$\dot{v} + 0.02v = u - 9.8\theta \Rightarrow \ddot{e} + 0.02\dot{e} = -\dot{u} + 9.8\dot{\theta} + \ddot{\bar{v}} + 0.02\dot{\bar{v}}$$
$$u(t) = k_p\left(\bar{v} - v(t)\right) + k_i \int_0^t \left(\bar{v} - v(\tau)\right) d\tau \Rightarrow \dot{u} = k_p\dot{e} + k_i e$$
$$\Downarrow$$
$$\boxed{\ddot{e} + \left(0.02 + k_p\right)\dot{e} + k_i e = 9.8\dot{\theta} + \dot{\bar{v}} + 0.02\bar{v}}$$

$$E(s) = \frac{(s + 0.02)}{s^2 + \left(0.02 + k_p\right)s + k_i}\bar{V}(s) + \frac{9.8s}{s^2 + \left(0.02 + k_p\right)s + k_i}\Theta(s)$$

Key issue: How do control parameters k_p, k_i affect performance?

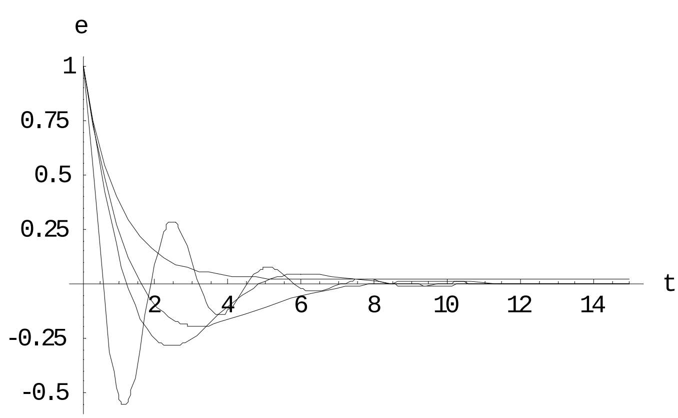


Figure 3. Error response to unit step command $k_p = 1, k_i = 0, 0.5, 1, 2, 6$.

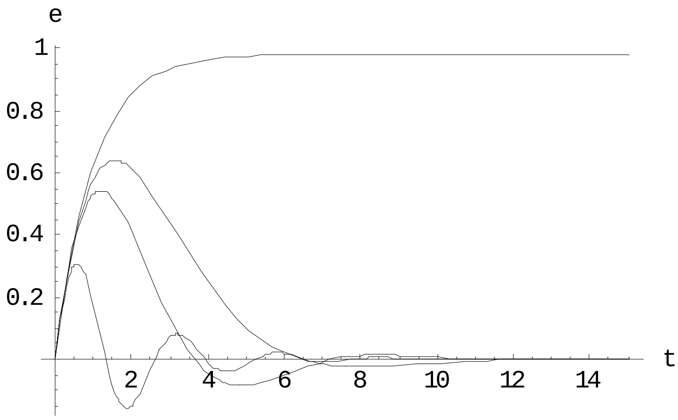
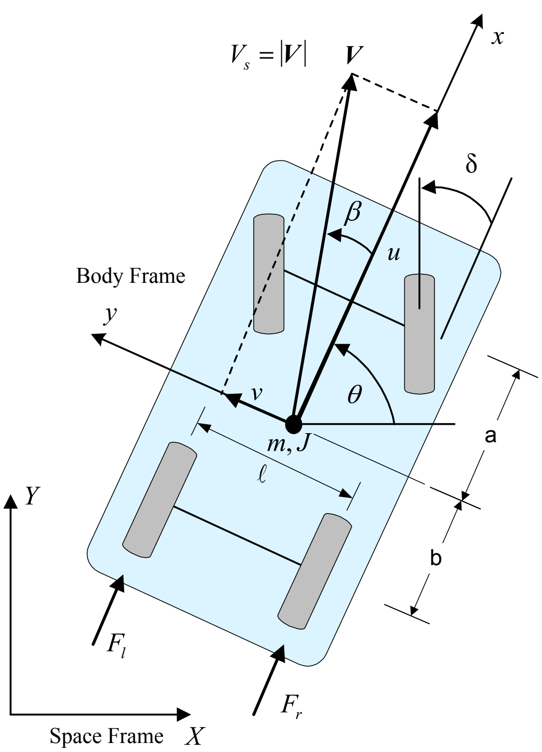


Figure 4. Error response to unit step disturbance $k_p = 1, k_i = 0, 0.5, 1, 2, 6$.

Automobile Directional Stability



The vehicle is assumed to travel at constant speed, V, and all tire cornering coefficients are the same, κ. Consider the transfer function $\delta \to \omega$

$$G(s) = 0.2756 \frac{s + 0.02576\kappa / V}{s^2 + (0.04939\kappa / V)s + \left(0.0006076\kappa^2 / V^2 - 0.0002428\kappa\right)}$$

The system transfer function is dependent on two parameters. For simplicity we will eliminate one of them. Suppose $\kappa = 6964.2$, so that

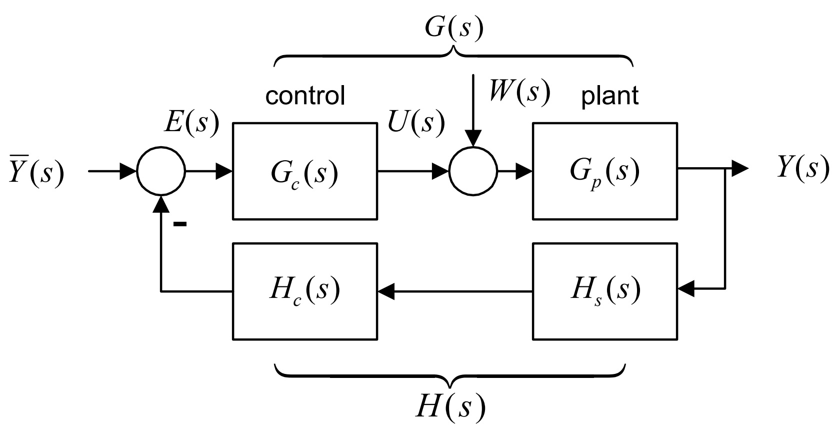
$$G(s) = 0.2756\frac{s + 179.4 / V}{s^2 + (344.0 / V)s + \left(29457.0 / V^2 - 1.691\right)}$$

Notice that the system is stable for low values of speed and unstable for large values. The poles are

$$s = \frac{1}{2V}\left[-344.\pm 2.6\sqrt{75.1035 + V^2}\right]$$

Problem Definition

The control system design considered here will the fixed control structure



The signals illustrated are: $\overline{Y}(s)$- command, $Y(s)$-output, $U(s)$-control, $E(s)$-error (this is a true error only if $H(s) = 1$), $W(s)$-disturbance

Here the plant transfer function, $G_p(s)$, is fixed. The series compensator transfer function, $G_c(s)$, is a free design choice. The feedback transfer function, $H(s)$, may be free in part, $H_c(s)$, and fixed in part ($H_s(s)$, non-negligible sensor dynamics).

Ordinarily, we will consider only two feedback compensators:

$$H_c(s) = \begin{cases} 1 & output\ feedback \\ 1 + c\,s & output + velocity\ feedback \end{cases}$$

We will consider several different series compensators, the most commonly used are listed in the following table.

$G_c(s)$	Name	Effect on Ultimate State Error	Effect on Stability
K	P (uncompensated)		
$K\dfrac{s+\alpha}{s}$	PI	Improves	Degrades
$K\dfrac{s^2+\alpha_1 s+\alpha_0}{s}$	PID	Improves	Improves somewhat
$K\dfrac{s+\alpha}{s+\beta}, \alpha>\beta$	lag	Improves somewhat	Degrades somewhat
$K\dfrac{s+\alpha}{s+\beta}, \alpha<\beta$	lead	Degrades somewhat	Improves somewhat
$K\left(s+\alpha\right)$	Rate feedback (PD)	Degrades	Improves
$K\dfrac{s^2+2\rho_1\omega_1 s+\omega_1^2}{s^2+2\rho_2\omega_2 s+\omega_2^2}$	Notch		Neutralizes plant resonance

Table 1. Standard series compensators.

The output and error responses to the command signal are

$$Y(s)=\frac{G(s)}{1+G(s)H(s)}\bar{Y}(s), \qquad E(s)=\frac{1}{1+G(s)H(s)}\bar{Y}(s)$$

The goals of control system design are three fold:

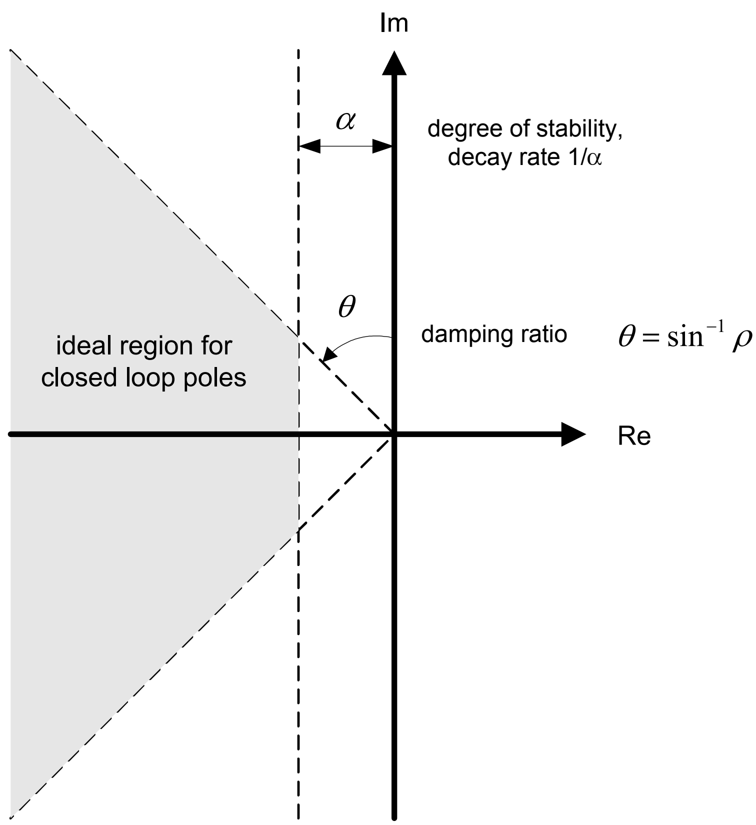
1. regulation, the output should track the command, $y(t)\to 0\pm\varepsilon$ as $t\to\infty$
2. transient dynamics, the closed loop poles should be located in a desirable region of the left half plane.
3. robustness, the closed loop stability/performance should be insensitive to model errors.

Regulation: Ultimate State Tracking Errors

Transient Dynamics

There are at least three ways to characterize the dynamic behavior of linear systems:

1. time domain (output time trajectories)
2. pole (or eigenvalue) location
3. frequency domain (Bode or Nyquist plots)



Root Locus Summary

The root locus method is a graphical procedure introduced in the mid- 1950's that helps choose controller parameters to locate closed loop poles in a desired region of the complex plane.

Problem: generate a sketch in the complex plane of the roots of the polynomial

$$d(s) + Kn(s) = 0\,, \quad \deg d(s) \ge \deg n(s)$$

as a function of the parameter K.

$$d(s) = (s - p_1)(s - p_2)\ldots(s - p_n)\,,$$

$$n(s) = (s - z_1)(s - z_2)\ldots(s - z_m)$$

Approach:

This is equivalent to finding the roots of

$$K\frac{n(s)}{d(s)} = -1 = e^{j(2k+1)\pi}\,,\ k = 0, \pm 1, \pm 2, \ldots \Rightarrow$$

$$\left| K\frac{n(s)}{d(s)} \right| = 1\,, \quad \textbf{magnitude condition} \Rightarrow K = \left| \frac{d(s)}{n(s)} \right|, \quad K \ge 0$$

$$\angle\left(K\frac{n(s)}{d(s)} \right) = (2k+1)\pi\,, \textbf{ angle condition} \Rightarrow \angle\left(\frac{n(s)}{d(s)} \right) = (2k+1)\pi, \quad K \ge 0$$

One strategy is to use the angle condition to locate the loci of roots and then to use the magnitude condition to calibrate the loci with respect to K.

Basic Rules

- **Number of branches**: The number of branches of the root locus equals the number of open loop poles.
- **Symmetry**: The root locus is symmetric about the real axis.

- **Starting & ending points**: The root locus begins at the open loop poles and ends at the finite and infinite open loop zeros.
- **Real-axis segments**: For $K > 0$, real axis segments to the left of an odd number of finite real axis poles and/or zeros are part of the root locus.
- **Behavior at infinity**: The root locus approaches infinity along asymptotes with angles:

$$\theta = \frac{(2k+2)\pi}{\#\, finite\ poles - \#\, finite\ zeros},\ k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

  Furthermore, these asymptotes intersect the real axis at a common point given by

$$\sigma = \frac{\sum finite\ poles - \sum finite\ zeros}{\#\, finite\ poles - \#\, finite\ zeros}$$

- **Real axis breakaway and break-in points**: The root locus breaks away from the real axis where the gain is a (local) maximum on the real axis, and breaks into the real axis where it is a local minimum. To locate candidate break points solve

$$\frac{d}{ds}\left(\frac{1}{GH(s)}\right) = 0$$

- $j\omega$ **-axis crossings**: Use Routh test to determine values of K for which loci cross imaginary axis.

Derivation of 'behavior at infinity'

First, recall the closed loop characteristic equation:

$$d(s) + Kn(s) = 0 \Rightarrow \frac{d(s)}{n(s)} = -K$$

Now, we are interested in the situation where s is a very large complex number. It is easier to consider the very small complex number $\varepsilon = 1/s$, so rewriting in terms of ε, we obtain the approximations

$$
\begin{aligned}
d(s) &= s^n + b_{n-1}s^{n-1} + \cdots + b_0 \\
&= \frac{1}{\varepsilon^n}\left(1 + b_{n-1}\varepsilon + \cdots + b_0\varepsilon^n\right) \\
&= \frac{1}{\varepsilon^n}\left(1 + b_{n-1}\varepsilon + O(\varepsilon^2)\right)
\end{aligned}
$$

$$n(s) = \frac{1}{\varepsilon^m}\left(1 + a_{m-1}\varepsilon + O(\varepsilon^2)\right)$$

Now, back to *s*,

$$\frac{d(s)}{n(s)} \approx s^{n-m}\ \frac{b_{n-1}+s}{a_{m-1}+s} = -K$$

$$s\left[1 + \frac{a_{m-1}-b_{n-1}}{(m-n)s} + O\left(\frac{1}{s^2}\right)\right] = (-K)^{1/(n-m)}$$

$$s + \frac{a_{m-1}-b_{n-1}}{(m-n)} = (-K)^{1/(n-m)}$$

$$s \to \sigma + \rho e^{j\theta}$$

$$\sigma + \rho e^{j\theta} + \frac{a_{m-1}-b_{n-1}}{(m-n)} = K^{1/(n-m)}e^{j(\pi+k2\pi)/(n-m)}$$

$$
\begin{aligned}
&\sigma = -\frac{a_{m-1}-b_{n-1}}{(m-n)} \\
&\rho = K^{1/(n-m)} \\
&\theta = (2k+1)\pi/(n-m)
\end{aligned}
$$

$$a_{m-1} = z_1 + \cdots + z_m,\ b_{n-1} = p_1 + \cdots + p_n$$

Additional Rules

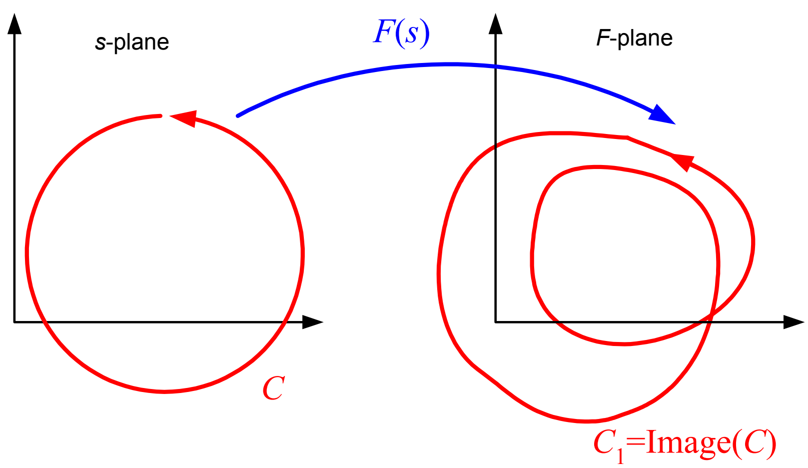
- **Angles of departure & arrival**:

Nyquist Method

Nyquist analysis is a graphical method that enables determination of closed loop stability from the open loop transfer function. But far more important is the fact that it allows identification of a 'stability margin,' that is, how much deformation of the open loop transfer function can be tolerated before the system becomes unstable.

Basics

Consider a mapping $F(s)$ that takes complex numbers from the s-plane into complex numbers in the F-plane. A simple closed curve C in the s-plane can be mapped into the F-plane to produce the closed curve $C_1 = \text{Image}(C)$.



Definition: A point a in the F-plane is encircled m times by a closed contour C_1 if the phasor F-a sweeps out an angle $2\pi m$ as s traverses C once in the positive direction.

Theorem (Cauchy): Principle of the argument. Let C be a simple closed curve in the s-plane. $F(s)$ is a rational function having neither poles nor zeros on C. If C_1 is the image of C under F, then

$$N = Z - P$$

where

N is the number of positive encirclements of the origin by C_1, as s traverses C one time in the positive direction.

Z is the number of zeros of F(s) enclosed by C, counting multiplicities,

P is the number of poles of F(s) enclosed by C, counting multiplicities.

Proof:

Nyquist makes three key innovations to derive a stability criterion from Cauchy's Theorem:

1. Take $F(s)=1+G(s)H(s)$ (or, $1+KG(s)H(s), \dfrac{1}{K}+G(s)H(s)$)
2. Define the Nyquist Contour in the s-plane to include
    a. Entire imaginary axis, avoiding poles of F(s),
    b. Infinite semi-circle enclosing the RHP
3. Shift, by –1, from F-plane to GH-plane

Theorem (Nyquist Criterion) If the Nyquist plot of $GH(s)$ (i.e., the image of the Nyquist contour in the *GH*-plane under one positive traverse of *C* encircles the point -1+*j*0 in the negative direction as many times as there are unstable open loop poles (poles of *GH* within the Nyquist contour) then the feedback system has no unstable poles.

First Examples

Gain & Phase Margin

Performance: Sensitivity Peaks & Bandwidth

The Sensitivity Functions

Sensitivity function: $S := [I+L]^{-1}$

Complementary sensitivity function: $T := [I+L]^{-1} L$

Consider a scalar system in which $L = GK$ is the open loop transfer function and $T=[1+L]^{-1}L$ is the closed loop transfer function. Then compute the (relative) variation of the closed loop with respect to (relative) variation of the open loop transfer function:

$$\frac{dT/T}{dL/L} = \frac{dT}{dL}\frac{L}{T}$$
$$= \left\{ -[1+L]^{-2} L + [1+L]^{-1} \right\} \frac{L}{[1+L]^{-1} L}$$
$$= -[1+L]^{-1} L + 1$$
$$= [1+L] = S$$

This is Bode's original reason for the terminology 'sensitivity function' for *S*.

Bode Waterbed Formula

Application of the Cauchy Integral Formula to systems with relative degree 2 or greater:

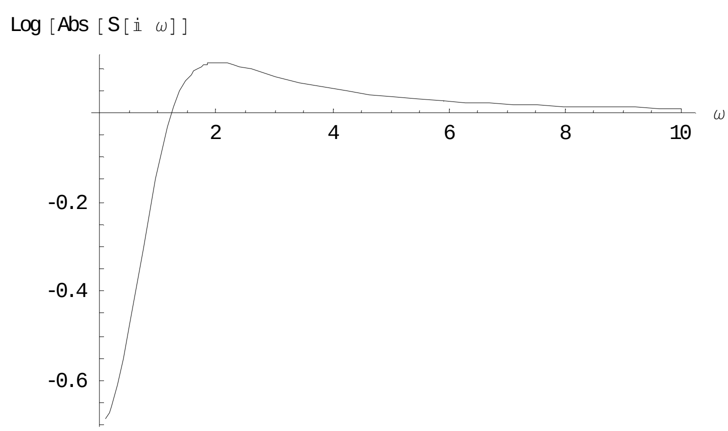
(Waterbed effect)

$$\int_0^\infty \ln|S(j\omega)|\, d\omega = \pi \sum_{ORHP\ poles} p_i$$

$$\int_0^\infty \ln|T(j\omega)| \frac{d\omega}{\omega^2} = \pi \sum_{ORHP\ zeros} \frac{1}{q_i}$$

Example: Stable plant

$$L(s) = \frac{1}{(s+1)^2}$$

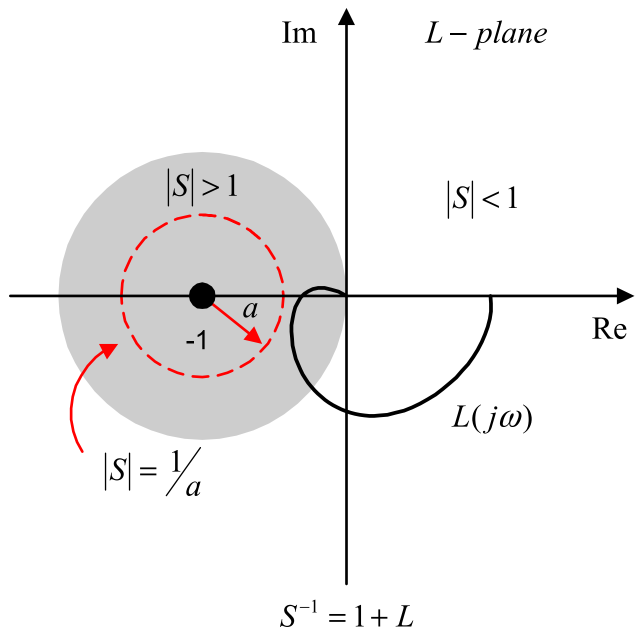


Sensitivity Peaks

$$M_S = \max_\omega |S(j\omega)|, \quad M_T = \max_\omega |T(j\omega)|$$

Sensitivity peaks are related to gain and phase margin.

Sensitivity peaks are related to overshoot.



Bandwidth

1. Bandwidth (sensitivity) $\omega_{BS} = \max_v \left\{ v : |S(j\omega)| < 1/\sqrt{2} \quad \forall \omega \in [0, v) \right\}$
2. Bandwidth (complementary sensitivity) $\omega_{BT} = \min_v \left\{ v : |S(j\omega)| < 1/\sqrt{2} \quad \forall \omega \in (v, \infty) \right\}$
3. Crossover frequency $\omega_c = \max_v \left\{ v : |L(j\omega)| \geq 1 \quad \forall \omega \in [0, v) \right\}$

Bandwidth is related to rise time and settling time.

Example

$$G(s) = \frac{0.2(s+1)(s^2+12)}{s(s^2+2s+2)(s^2+2s+4)}$$

$$G[s] = \frac{0.2\,(s+1)\,(s^2+12)}{s\,(s^2+2s+2)\,(s^2+2s+4)}$$

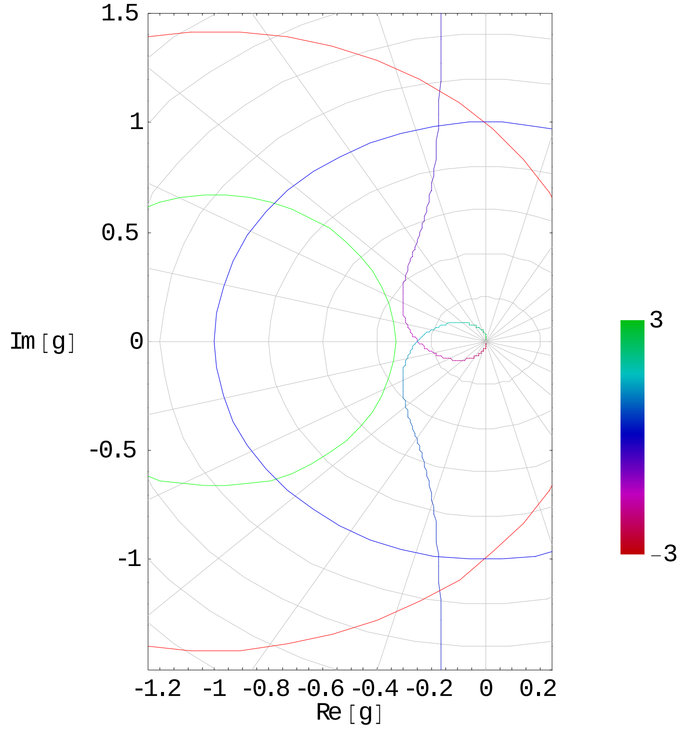


Figure 5. The green circle of radius 1/1.5 corresponds to a sensitivity function peak of 1.5. The unit circle about the origin (blue) allows identification of the phase margin. The red circle of radius $\sqrt{2}$ identifies the (sensitivity function) bandwidth.

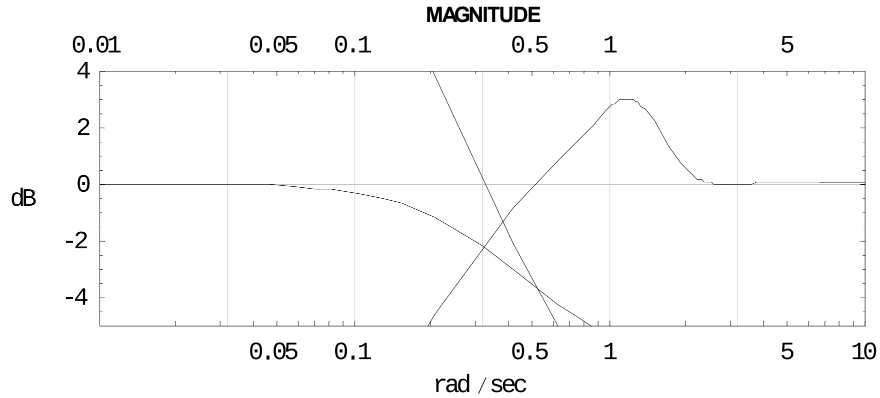


Figure 6. The three definitions of bandwidth can be compared for this example: 'sensitivity function,' 'complementary sensitivity function,' and 'crossover frequency.' In this figure which is which?