DYNAMICS AND SIMULATION OF THE SIMPLEST MODEL OF A SKATEBOARD

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Abstract
In the present paper analysis and simulation are performed for the simplest model of a skateboard. We suppose the skateboard is uncontrollable. The equations of motion of the model are derived and their stability analysis is fulfilled.

Key words
skateboard; nonholonomic constraints; integrability; normal form; simulation

1 Introduction

Nowadays skateboarding that is rider's skill, has become one of the most popular kind of sport. Nevertheless serious researches concerning dynamics and stability of a skateboard are almost absent. At the late 70th - early 80th of the last century Mont Hubbard [1; 2] proposed the two mathematical models describing motions of a skateboard in the presence of a rider. To derive the equations of motion he used the principal theorems of dynamics. In our paper we give the further development of the models proposed by Hubbard.

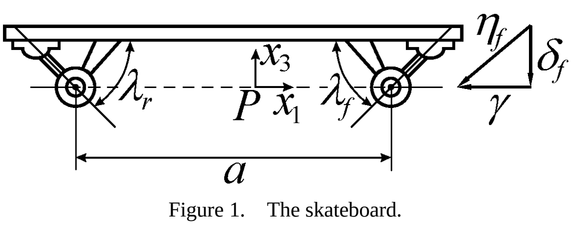

Figure 1. The skateboard.

A skateboard typically consists of a board, two trucks and four wheels (Fig. 1). The modern boards are usually from 78 to 83 cm long, 17 to 21 cm wide and 1 to 2 cm thick. The most essential elements of the skateboard are the trucks, connecting the axles to the board. Angular motion of both the front and rear axles is constrained to be about their respective nonhorizontal pivot axes, thus causing a steering angle of the wheels whenever the axles are not parallel to the plane of the board (Fig. 1-2). The vehicle is steered by making use of this kinematic relationship between steering angles and tilt of the board. In addition, there is a torsional spring, which exerts the restoring torque between the wheelset and the board proportional to the tilt of the board with respect to the wheelset (Fig. 3). We denote the stiffness of this spring by k_1.

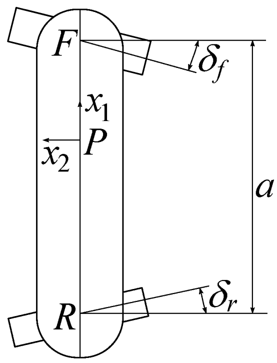

Figure 2.

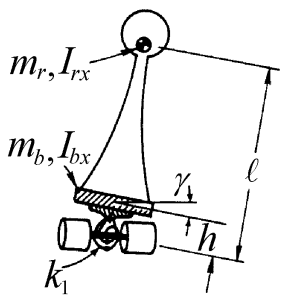

Figure 3.

2 Formulation of the Problem. Equations of Motion

We assume that a rider is modeled with a rigid bar that is perpendicular relative to the board. Therefore, when the board tilts through γ, the rider tilts through the same angle relative to the vertical. Let us introduce an inertial coordinate system $OXYZ$ in the ground plane. Let $FR = a$ be the distance between the two axle centers F and R of the skateboard. The position of the line FR with respect to the $OXYZ$-system is defined by X and Y coordinates of its centre and by the angle θ between this line and the OX-axis (Fig. 4).

The tilt of the board is accompanied by rotation of the front wheels clockwise through δ_f and rotation of the rear wheels anticlockwise through δ_r (Fig. 2, 4). The wheels of the skateboard are assumed to roll without lateral sliding. This condition is modeled by the con-

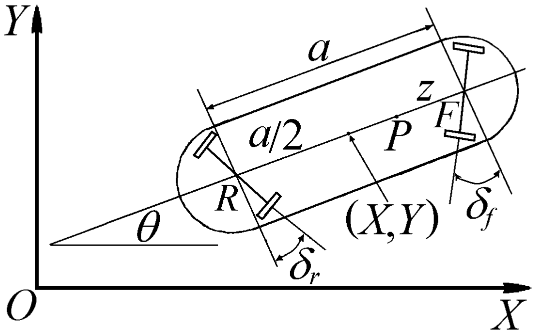

Figure 4. The basic coordinate systems.

straints which are nonholonomic as can be proved

$$\dot{Y}\cos(\theta-\delta_f)-\dot{X}\sin(\theta-\delta_f)+\frac{a}{2}\,\dot{\theta}\cos\delta_f=0,$$
$$\dot{Y}\cos(\theta+\delta_r)-\dot{X}\sin(\theta+\delta_r)-\frac{a}{2}\,\dot{\theta}\cos\delta_r=0. \quad (1)$$

Under these conditions velocities of the points F and R will be directed horizontally and perpendicularly to the axles of the wheels and there is the point P on the line FR which has zero lateral velocity. We denote its forward velocity by u. It may be shown, that (see e.g. [1]-[6])

$$u=-\frac{a\dot{\theta}\cos\delta_f\cos\delta_r}{\sin(\delta_f+\delta_r)},$$

$$FP=\frac{a\sin\delta_f\cos\delta_r}{\sin(\delta_f+\delta_r)},\ \dot{\theta}=-\frac{u\sin(\delta_f+\delta_r)}{a\cos\delta_f\cos\delta_r}. \quad (2)$$

Using the results obtained in [5; 6] we conclude that the steering angles δ_f and δ_r are related to the tilt of the board by the following equations

$$\tan\delta_f=\tan\lambda_f\sin\gamma,\quad \tan\delta_f=\tan\lambda_f\sin\gamma, \quad (3)$$

where λ_f and λ_r are the fixed angles which the front and rear axes make with the horizontal (Fig. 1). Using constraints (3) we can rewrite equations (1) as follows:

$$\dot{X}=u\cos\theta+\frac{(\tan\lambda_f-\tan\lambda_r)}{2}u\sin\gamma\sin\theta,$$
$$\dot{Y}=u\sin\theta-\frac{(\tan\lambda_f-\tan\lambda_r)}{2}u\sin\gamma\cos\theta. \quad (4)$$

Expressions (2) become

$$FP=\frac{a\tan\lambda_f}{\tan\lambda_f+\tan\lambda_r},\ \dot{\theta}=-\frac{(\tan\lambda_f+\tan\lambda_r)}{a}u\sin\gamma. \quad (5)$$

Suppose that the board of the skateboard is located a distance h above the line FR. The length of the board is also equal to a. The board center of mass is located in its center. As to the rider we suppose that rider's center of mass is not located above the board center of mass, but it is located over the central line of the board a distance d from the front truck. Let l be the height of the rider's center of mass above the point P. Other parameters for the problem are: m_b is mass of the board; m_r is mass of the rider; I_{bx}, I_{by}, I_{bz} are the principal central moments of inertia of the board; I_{rx}, I_{ry}, I_{rz} are the principal central moments of inertia of the rider. We introduce also the following parameters:

$$I_x=I_{bx}+I_{rx},\quad I_y=I_{by}+I_{ry},\quad I_z=I_{bz}+I_{rz}.$$

It can be proved (see [5]) that the variables u and γ satisfy the following differential equations

$$\left(A+(C-2D)\sin^2\gamma+K\sin^4\gamma\right)\dot{u}+$$
$$+\left(C-3D+3K\sin^2\gamma\right)u\dot{\gamma}\sin\gamma\cos\gamma+$$
$$+B\left(\ddot{\gamma}\cos\gamma-\dot{\gamma}^2\sin\gamma\right)\sin\gamma=0,$$
$$E\ddot{\gamma}+\left(D-K\sin^2\gamma\right)u^2\sin\gamma\cos\gamma+ \quad (6)$$
$$+k_1\gamma-(m_bh+m_rl)\,g\sin\gamma+$$
$$+B\left(\dot{u}\sin\gamma+u\dot{\gamma}\cos\gamma\right)\cos\gamma=0.$$

Here A, ..., E, K – are the functions of the parameters, namely

$$A=m_b+m_r,\ E=I_x+m_bh^2+m_rl^2,$$

$$B=\frac{m_bh}{2}\left(\tan\lambda_f-\tan\lambda_r\right)+$$
$$+\frac{m_rl}{a}\left((a-d)\tan\lambda_f-d\tan\lambda_r\right),$$

$$C=\frac{m_b}{4}\left(\tan\lambda_f-\tan\lambda_r\right)^2+\frac{I_z}{a^2}\left(\tan\lambda_f+\tan\lambda_r\right)^2$$
$$+\frac{m_r}{a^2}\left((a-d)\tan\lambda_f-d\tan\lambda_r\right)^2,$$

$$D=\frac{(\tan\lambda_f+\tan\lambda_r)}{a}\left(m_bh+m_rl\right),$$

$$K=\frac{(\tan\lambda_f+\tan\lambda_r)^2}{a^2}\left(I_y+m_bh^2+m_rl^2-I_z\right).$$

Thus, equations (4)-(6) form the close system of equation of the skateboard motion.

3 Stability of the Straight-line motion of the Skateboard

Equations (6) have the particular solution

$$u = u_0 = \mathrm{const}, \quad \gamma = 0, \tag{7}$$

which corresponds to uniform straight-line motion of the skateboard. The conditions of stability for this particular solution have the following form [1]-[6]:

$$Bu_0 > 0, \quad Du_0^2 + k_1 - (m_b h + m_r l)\, g > 0 \tag{8}$$

From the first condition (8) we can conclude that the stability of motion (7) depends on its direction. If the motion in one direction is stable then the motion in the opposite direction is necessary unstable. Such behavior is peculiar to many nonholonomic systems. First of all, we can mention here the classical problem of motion of a rattleback (aka wobblestone or celtic stone, see e.g. [7]-[9]). In this problem the stability of permanent rotations of a rattleback also depends on the direction of rotation.

Suppose that the coefficient B is positive, $B > 0$. Then for $u_0 > 0$ the skateboard moves in the "stable" direction, and for $u_0 < 0$ it moves in the "unstable" direction. When $u_0 = 0$ the skateboard is in equilibrium position on the plane. The necessary and sufficient condition for the stability of this equilibrium have the form [1]-[6]:

$$k_1 - (m_b h + m_r l)\, g > 0. \tag{9}$$

Assuming that condition (9) holds, let us consider the behaviour of the system near the equilibrium position. Solving equations (6) with respect to $\dot u$ and $\ddot\gamma$ and assuming that u, γ and $\dot\gamma$ are small, we can write the equations of the perturbed motion taking into account the terms which are quadratic in u, γ and $\dot\gamma$:

$$\dot u = \frac{B\Omega^2}{A}\gamma^2, \quad \ddot\gamma + \Omega^2\gamma = -\frac{Bu\dot\gamma}{E}, \tag{10}$$

where we introduce the following notation

$$\Omega^2 = \frac{k_1 - (m_b h + m_r l)\, g}{E}.$$

Note, that the linear terms in the second equation of the system (10) have the form which corresponds to normal oscillations. To examine the nonlinear system (10) we reduce it to normal form [10]. To obtain the normal form of the system (10) first of all we replace the variables with the two complex-conjugate variables z_1 and z_2:

$$\gamma = \frac{z_1 - z_2}{2i}, \ \dot\gamma = \frac{z_1 + z_2}{2}\Omega, \ u = z_3.$$

In the variables z_k, $k = 1, 2, 3$ the linear part of the system (10) has the diagonal form and the derivation of its normal form reduces to separating of resonant terms from the nonlinearities in the right-hand sides of the transformed system (10). Finally, the normal form of the system (10) may be written as follows:

$$\dot z_1 = i\Omega z_1 - \frac{B}{2E}z_1 z_3, \ \dot z_2 = -i\Omega z_2 - \frac{B}{2E}z_2 z_3,$$
$$\dot z_3 = \frac{B\Omega^2}{2A}z_1 z_2.$$

Introducing the real polar coordinates according to the formulae

$$z_1 = \rho_1(\cos\sigma + i\sin\sigma), \ z_2 = \rho_1(\cos\sigma - i\sin\sigma),$$
$$z_3 = \rho_2$$

we obtain from the system (10) the normalized system of equations of the perturbed motion which is then split it into two independent subsystems:

$$\dot\rho_1 = -\frac{B}{2E}\rho_1\rho_2, \quad \dot\rho_2 = \frac{B\Omega^2}{2A}\rho_1^2, \tag{11}$$

$$\dot\sigma = \Omega. \tag{12}$$

Terms of order higher than the second in (11) and those higher than the first in ρ_k, $k = 1, 2$ in (12) have been omitted here.

In the ε-neighborhood of the equilibrium position the right-hand sides of equations (11) and (12) differ from the respective right-hand sides of the exact equations of the perturbed motion by quantities of order ε^3 and ε^2 respectively. The solutions of the exact equations are approximated by the solutions of system (11)-(12) with an error of ε^2 for ρ_1, ρ_2 and of order ε for σ in the time interval of order $1/\varepsilon$. Restricting the calculations to this accuracy, we will consider the approximate system (11)-(12) instead of the complete equations of the perturbed motion.

The equation (12) is integrable. We obtain

$$\sigma = \Omega t + \sigma_0.$$

System (11) describes the evolution of the amplitude ρ_1 of the board oscillations and also the evolution of the velocity ρ_2 of the straight-line motion of the skateboard. One can see that this system has the first integral

$$E\rho_1^2 + \frac{A}{\Omega^2}\rho_2^2 = An_1^2, \tag{13}$$

where n_1 is the constant, specified by initial conditions. We will use this integral for solving of the system (11)

and for finding the variables ρ_1 and ρ_2 as functions of time: $\rho_1 = \rho_1(t)$, $\rho_2 = \rho_2(t)$. Expressing ρ_1^2 from the integral (13) and substitute it to the second equation of the system (11) we get

$$\dot{\rho}_2 = \frac{B}{2E}\left(\Omega^2 n_1^2 - \rho_2^2\right). \tag{14}$$

The general solution of equation (14) has the following form:

$$\rho_2(t) = \frac{\Omega n_1\left(1 - n_2 \exp\left(-\frac{B\Omega n_1}{E}t\right)\right)}{\left(1 + n_2 \exp\left(-\frac{B\Omega n_1}{E}t\right)\right)}, \tag{15}$$

where n_2 is a nonnegative arbitrary constant. Now, using the integral (13), we can find the explicit form of the function $\rho_1(t)$:

$$\rho_1(t) = 2n_1\sqrt{\frac{An_2}{E}}\frac{\exp\left(-\frac{B\Omega n_1}{2E}t\right)}{1 + n_2 \exp\left(-\frac{B\Omega n_1}{E}t\right)}. \tag{16}$$

Let us consider the properties of the solutions (15), (16) of system (11) and their relations with the properties of motion of the skateboard. System (11) has the equilibrium position

$$\rho_1 = 0, \quad \rho_2 = \Omega n_1 \tag{17}$$

(these particular solutions can be obtained from the general functions (15)-(16) if we suppose in these functions $n_2 = 0$). The arbitrary constant n_1 can take any sign. The positive values of this constant correspond to the straight-line motions of the skateboard with a small velocity in the "stable" direction and the negative values correspond to the "unstable" direction. Indeed, if we linearize equations (11) near the equilibrium position (17) we get

$$\dot{\rho}_1 = -\frac{B}{2E}\Omega n_1 \rho_1, \ \dot{\rho}_2 = 0.$$

Thus, for $n_1 > 0$ the equilibrium position (17) is stable and for $n_1 < 0$ it is unstable.

Varying the functions ρ_1 and ρ_2 we obtain the behaviour of the skateboard moving with small velocities. Let us suppose, that at the initial instant the system is near the stable equilibrium position $(n_1 > 0)$ and $\rho_2(0) \geq 0$, i.e. $n_2 \leq 1$ (the case when $n_1 > 0, n_2 > 1$ is similar to the case $n_1 < 0, n_2 < 1$, which will be investigated below). These initial conditions correspond the situation when at the initial instant the skateboard takes the small velocity

$$\rho_2(0) = \Omega n_1 \frac{1 - n_2}{1 + n_2}$$

in the "stable" direction. Then in the course of time the "amplitude" of oscillations of the board ρ_1 decreases monotonically from its initial value

$$\rho_1(0) = \frac{2n_1}{1 + n_2}\sqrt{\frac{An_2}{E}}$$

to zero, while the magnitude of the skateboard velocity ρ_2 increases. In the limit the skateboard moves in the "stable" direction with the constant velocity Ωn_1 (see Fig. 5-6).

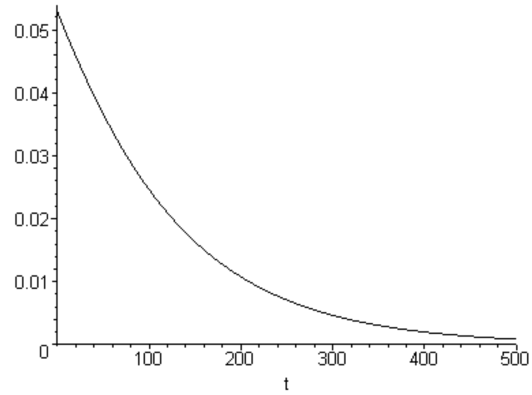

Figure 5. Evolution of the amplitude ρ_1 of the board oscillations in time for the case $n_1 > 0, n_2 \leq 1$.

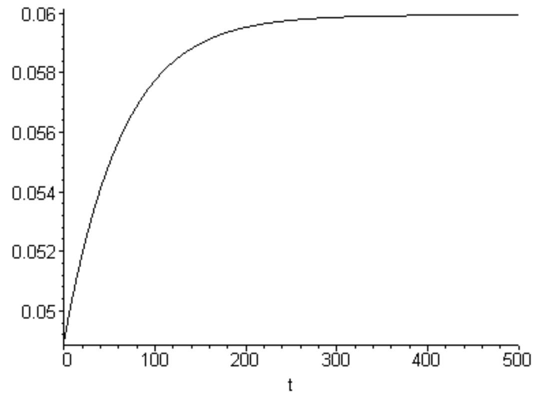

Figure 6. Evolution of the "velocity" ρ_2 of the skateboard in time for the case $n_1 > 0, n_2 \leq 1$.

Suppose now that at the initial instant the system is near the unstable equilibrium position $n_1 < 0$. Suppose again, that at the initial instant $n_2 < 1$, i.e. $\rho_2(0) < 0$ (the case $n_1 < 0$, $n_2 > 1$ is similar to the case $n_1 > 0$, $n_2 < 1$ which was considered above). These initial conditions correspond the situation when at the initial instant the skateboard takes the small velocity

$$\rho_2(0) = \Omega n_1 \frac{1 - n_2}{1 + n_2}$$

in the "unstable" direction. In this case the limit of the system motions is the same as when $\rho_2(0) \geq 0$ but the evolution of the motion is entirely different. When

$$0 < t < t_* = \frac{E \ln(n_2)}{B\Omega n_1}$$

the absolute value of the oscillation "amplitude" ρ_1 increases monotonically and the skateboard moves in the "unstable" direction with decreasing velocity. At the instant $t = t_*$ the velocity vanishes and the oscillation "amplitude" ρ_1 reaches its maximum absolute value

$$\rho_1(t_*) = n_1\sqrt{\frac{A}{E}}.$$

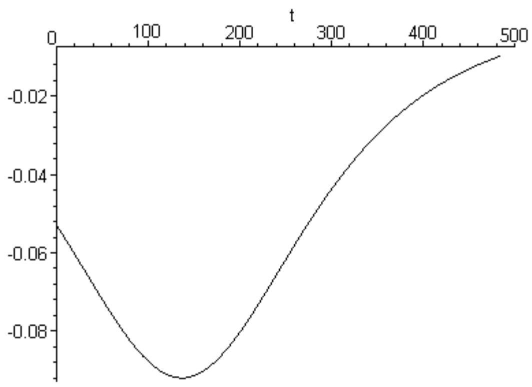

Figure 7. Evolution of the amplitude ρ_1 of the board oscillations in time for the case $n_1 < 0, n_2 \leq 1$.

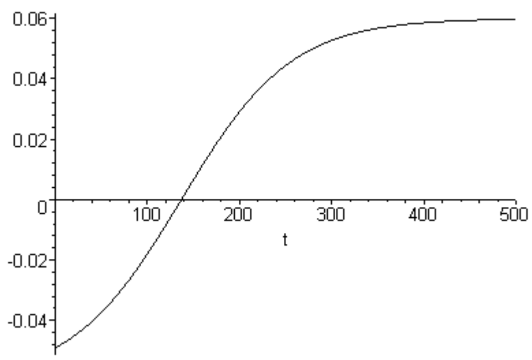

Figure 8. Evolution of the "velocity" ρ_2 of the skateboard in time for the case $n_1 < 0, n_2 \leq 1$.

When $t > t_*$ the skateboard already moves in the "stable" direction with increasing magnitude of its velocity and the oscillation amplitude decreases monotonically. Thus when $\rho_2(0) < 0$ the skateboard changes the direction of its motion (Fig. 7-8). The similar nonlinear effects (in particular the change of the direction of motion) were observed earlier in other problems of nonholonomic mechanics (for example in the classical problem of dynamics of a rattleback [7]-[9]). Thus, we describe here the basic features of dynamics of the simplest skateboard model, proposed in [1; 2] and developed by us.

4 Conclusions

In this paper the problem of motion of the skateboard with a rider was examined. This problem has many common features with other problems of nonholonomic dynamics. In particular it was shown that the stability of motion of the skateboard depends on the direction of motion. Moreover the system can change its direction of motion. The similar effects have been found earlier in the classical problem of a rattleback dynamics.

Acknowledgements

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