Flight Dynamics & Control

Equations of Motion of 6 dof Rigid Aircraft-Kinematics

Harry G. Kwatny

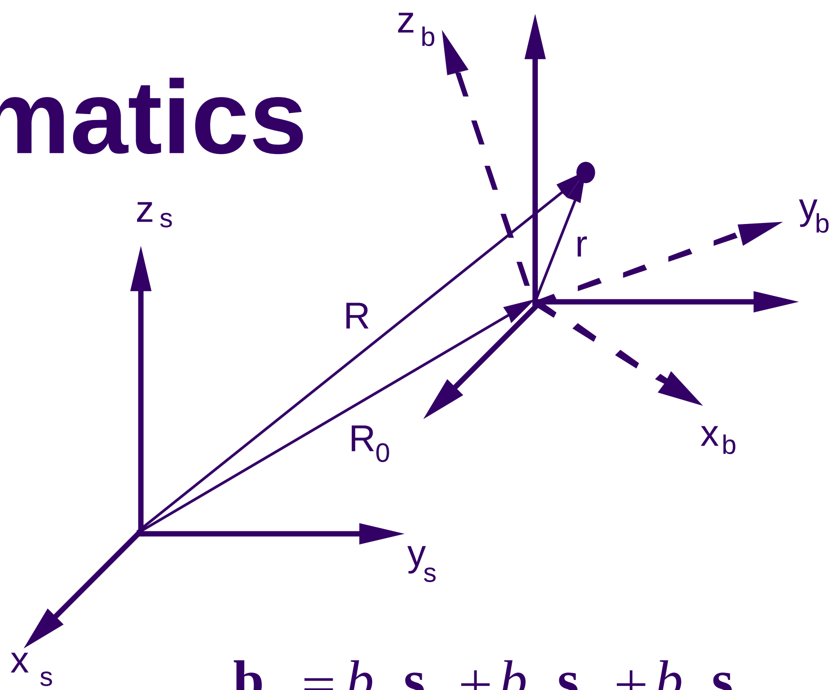
Department of Mechanical Engineering & Mechanics

Drexel University

Outline

- Rotation Matrix
- Angular Velocity
- Euler Angles
- Kinematic equations

Rigid Body kinematics



Consider two reference frames: a space frame (s) and a body frame (b) with common origin (R_0=0). Let {$\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$} be an orthonormal basis for the space frame. Let {$\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$} be an orthonormal basis for the body frame. The orientation of the body frame is specified relative to the space frame if the basis vectors {$\mathbf{b}_x$, $\mathbf{b}_y$, $\mathbf{b}_z$} are specified in the coordinates of the space frame.

$$\mathbf{b}_x = b_{xx}\mathbf{s}_x + b_{xy}\mathbf{s}_y + b_{xz}\mathbf{s}_z$$
$$\mathbf{b}_y = b_{yx}\mathbf{s}_x + b_{yy}\mathbf{s}_y + b_{yz}\mathbf{s}_z$$
$$\mathbf{b}_z = b_{zx}\mathbf{s}_x + b_{zy}\mathbf{s}_y + b_{zz}\mathbf{s}_z$$

Rotation Matrix

$$L := \begin{bmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{bmatrix}$$

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Properties of the Rotation Matrix

1) L^T converts body coordinates to space coordinates.

To see this, suppose a vector **r** has coordinates r_{bx}, r_{by}, r_{bz} in the body frame and r_{bx}, r_{by}, r_{bz} in the space frame:

$$\mathbf{r} = r_{bx}\mathbf{b}_x + r_{by}\mathbf{b}_y + r_{bz}\mathbf{b}_z = r_{sx}\mathbf{s}_x + r_{sy}\mathbf{s}_y + r_{sz}\mathbf{s}_z$$

multiply successively by $\mathbf{s}_x^T, \mathbf{s}_y^T, \mathbf{s}_z^T$ to prove

$$\begin{bmatrix} r_{sx} \\ r_{sy} \\ r_{sz} \end{bmatrix} = \begin{bmatrix} b_{xx} & b_{yx} & b_{zx} \\ b_{xy} & b_{yy} & b_{zy} \\ b_{xz} & b_{yz} & b_{zz} \end{bmatrix} \begin{bmatrix} r_{bx} \\ r_{by} \\ r_{bz} \end{bmatrix} \Leftrightarrow r^s = L^T r^b$$

2) Orthonormal unit vectors $\Rightarrow L^T L = I, L^T = L^{-1}$

3) Right hand coordinate system $\Rightarrow \det L = 1$

4) Translation plus rotation

$$R^s = R_0^s + L^T r^b, \quad \dot{R}^s = \dot{R}_0^s + \dot{L}^T r^b + L^T \dot{r}^b$$

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Successive Rotations

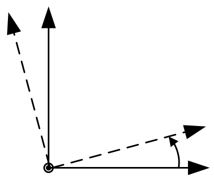
Suppose a succession of rotations are made,

say L_1, then L_2, then L_3, then the total rotation

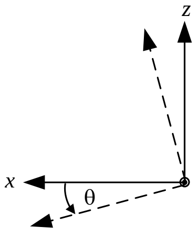
is defined by $L^T = L_3^T L_2^T L_1^T$.

Example:

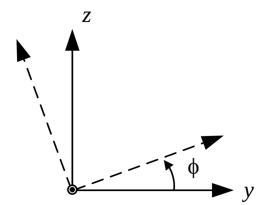
1) Rotation of angle ψ about z-axis $L_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$



2) Rotation of angle θ about y-axis $L_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$



3) Rotation of angle ϕ about x-axis $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$



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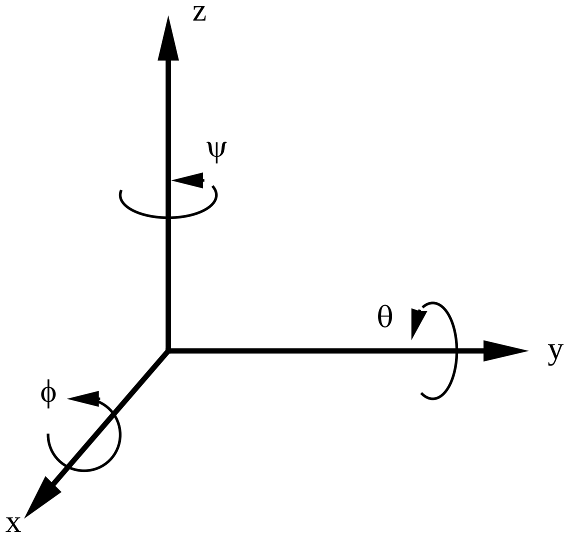
Euler Angles

Consider a reference frame fixed in the body, with origin located by the position vector R_0 and angular orientation denoted by L, both relative to a fixed inertial (space) frame.

L can be parameterized by the Euler angles ψ, θ, ϕ (yaw, pitch, roll) representing sequential rotations about the axes $z,\ y,\ x$, respectively:

$$L(\psi,\theta,\phi) = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

Standard coordinate frame employs 3,2,1 or z,y,x convention for defining Euler angles.



Angular Velocity ~ 1

Consider a rotation $L \to L + \Delta L$

Definition : $\dot{L} = \lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t}$

Definition : A square matrix A is called anti-symmetric if $A^T = -A$.

Example: A 3×3 anti-symmetric matrix has the general form

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

Notice that there are only 3 independent elements a, b, c. In this sense every 3×3 matrix is equal to a 3-vector. We use the notation

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad \tilde{v} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

Definition : The cross product of two 3-vectors u, v is $u \times v = \tilde{u} v$

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Angular Velocity ~ 2

Proposition : $L^T\dot{L}$ and $\dot{L}\,L^T$ and antisymmetric matrices.

Proof:

$$\left(L+\Delta L\right)^T\left(L+\Delta L\right)=L^TL+L^T\Delta L+\Delta L^TL+\Delta L^T\Delta L$$

but

$$L^TL=I,\left(L+\Delta L\right)^T\left(L+\Delta L\right)=I$$

$$\Rightarrow L^T\Delta L+\Delta L^TL+\Delta L^T\Delta L=0\Rightarrow L^T\dot{L}+\dot{L}^TL=0\Rightarrow L^T\dot{L}=-\left(L^T\dot{L}\right)^T$$

Definition : Define the antisymmetric matrices $\tilde{\omega}_s=\dot{L}^TL,\ \tilde{\omega}_b=L\dot{L}^T.\ \tilde{\omega}_s\sim\omega_s$, $\tilde{\omega}_b\sim\omega_b$ are the angular velocity in space and body coordinates, respectively.

Note: $\tilde{\omega}_s=L^T\tilde{\omega}_bL,\ \tilde{\omega}_b=L\tilde{\omega}_sL^T,\ \dot{L}^T=\tilde{\omega}_sL^T,\ \dot{L}=-\tilde{\omega}_bL$

$$\Rightarrow\dot{R}^s=\dot{R}_0^s+\dot{L}^Tr^b+L^T\dot{r}^b=\dot{R}_0^s+\tilde{\omega}_sL^Tr^b+L^T\dot{r}^b=\dot{R}_0^s+\omega_s\times L^Tr^b+L^T\dot{r}^b$$

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Velocity in Body and Space Coordinates

Consider a body frame (b) and a space frame (s) with common origin and the only relative motion is rotation. If **r** is the position vector of any point fixed in the body, then

$$r^s(t) = L^T(t) r^b,\; r^b = constant$$

$$v^s(t) = \dot{r}^s(t) = \dot{L}^T(t) r^b = \dot{L}^T(t) L r^s(t) = \tilde{\omega}_s(t) r^s(t) = \omega_s(t) \times r^s(t)$$

Similarly,

$$v^b(t) = L(t) v^s(t) = L(t)\tilde{\omega}_s(t) r^s(t) = L(t)\tilde{\omega}_s(t) L^T(t) r^b(t) = \tilde{\omega}_b(t) r^b(t) = \omega_b(t) \times r^b(t)$$

Translation + Rotation

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{r} \Leftrightarrow R^s = R_0^s + r^s \Leftrightarrow R^b = R_0^b + r^b$$

Inertial velocity in space coordinates

$$V^s = \dot{R}^s = \dot{R}_0^s + \dot{L}^T r^b = \dot{R}_0^s + \tilde{\omega}_s L^T r^b = \dot{R}_0^s + \omega_s \times L_{sb} r^b$$

Inertial velocity in body coordinates

$$V^b = L\dot{R}^s = L\dot{R}_0^s + L\dot{L}^T r^b = L\dot{R}_0^s + L\tilde{\omega}_s L^T r^b = L\dot{R}_0^s + \tilde{\omega}_b r^b = L\dot{R}_0^s + \omega_b \times r^b$$

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Euler Angle Kinematics

Recall the fundamental kinematic relationship: $\dot{L}(t) = -\tilde{\omega}_b(t)L(t)$

define the coordinate vector $q := \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$,

$$\dot{q} = \Gamma(q)\omega_b,$$

$$\Gamma(q) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix}, \Gamma^{-1}(q) = \begin{bmatrix} 1 & 0 & -\sin\phi \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Kinematic Equations Summary

inertial space location

$$\begin{bmatrix} \dot{x}_s \\ \dot{y}_s \\ \dot{z}_s \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\phi\cos\psi\sin\theta + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

angular orientation

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$