Flight Dynamics & Control

Equations of Motion of 6 dof Rigid Aircraft-Dynamics

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Outline

- Angular momentum
- Dynamical equations of motion
- Qualitative introduction to aero dynamic forces
- General forms of aerodynamic forces and moments

Angular Momentum ~ particle

Definition: Let O and P denote two points fixed in a rigid body and let $\mathbf{r}$ denote location of P relative to O. If $\mathbf{f}$ is a force applied at P, then the *moment of force* about O or *torque* about O is defined as

$$\mathbf{T} = \mathbf{r} \times \mathbf{f}$$

Definition: If a particle of mass m is located at P and the particle has (inertial) velocity $\mathbf{v}$ then its *moment of momentum* about O or *angular momentum* about O is defined as

$$\mathbf{H} = m\,\mathbf{r} \times \mathbf{v}$$

Angular Momentum ~ rigid body

Suppose O is the origin of a body fixed reference frame in a rigid body of volume $\mathcal{V}$ and mass distribution *dm*. Let $\mathbf{v}_0$ denote the (inertial) velocity of O and $\boldsymbol{\omega}$ the angular velocity of the rigid body. Then the angular momentum of the rigid body about O is defined as:

$$H = \int_{\mathcal{V}} \mathbf{r} \times \left[\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} \right] dm$$

Example: Torque in Body Coordinates

$$r^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; f^b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$$T^b = \tilde{r}^b f^b = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} -z\,f_y + y\,f_z \\ z\,f_x - x\,f_z \\ -y\,f_x + x\,f_y \end{bmatrix}$$

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Example: Rigid Body Angular Momentum in com Coordinates

$$r^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \omega^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \Rightarrow \tilde{r}^b \tilde{\omega}^b r^b = \begin{bmatrix} p\left(y^2+z^2\right) - qxy - rxz \\ -pxy + q\left(x^2+z^2\right) - ryz \\ -pxz - qyz + r\left(x^2+y^2\right) \end{bmatrix}$$

O is the com, so $\int_{\mathcal{V}} r^b dm = 0 \Rightarrow$

$$H^b = \int_{\mathcal{V}} \begin{bmatrix} p\left(y^2+z^2\right) - qxy - rxz \\ -pxy + q\left(x^2+z^2\right) - ryz \\ -pxz - qyz + r\left(x^2+y^2\right) \end{bmatrix} dm = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$I_x = \int_{\mathcal{V}} \left(y^2+z^2\right) dm \quad I_y = \int_{\mathcal{V}} \left(x^2+z^2\right) dm \quad I_z = \int_{\mathcal{V}} \left(x^2+y^2\right) dm$$

$$I_{xy} = \int_{\mathcal{V}} xy\, dm \quad I_{xZ} = \int_{\mathcal{V}} xz\, dm \quad I_{yz} = \int_{\mathcal{V}} yz\, dm$$

$$x - z \text{ symmetry} \Rightarrow I_{yz} = 0, I_{xz} = 0$$

Equations of Motion

Consider the motion of a rigid body. Let $\mathbf{v_0}$ denote its (inertial) velocity at its center of mass and $\mathbf{H_0}$ its angular momentum about its center of mass. Then Newton's second law states:

$$\frac{d}{dt}\left(m\mathbf{v_0}\right) = \sum \mathbf{F}$$

$$\frac{d}{dt}\mathbf{H_0} = \sum \mathbf{M}$$

Suppose that all vector quantities $\mathbf{v_0}, \mathbf{H_0}, \mathbf{F}, \mathbf{M}$ are specified in body coordinates, then

$$\frac{d}{dt}\left(mv_o^b\right) + m\tilde{\omega}^b v_o^b = \sum F^b$$

$$\frac{d}{dt}H_0^b + \tilde{\omega}^b H_0^b = \sum M^b$$

Now, $H_0^b = I^b \omega^b$. Assume m, I are constant, drop sub-suprscripts:

$$\boxed{\begin{aligned} m\dot{v} + m\tilde{\omega}v &= \sum F \\ I\dot{\omega} + I\tilde{\omega}\omega &= \sum M \end{aligned}}$$

Aircraft Dynamics in Euler Angles

Force Equations

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = m\begin{bmatrix} rv - qw \\ pw - ru \\ qu - pv \end{bmatrix} + mg\begin{bmatrix} -sin\theta \\ cos\theta sin\phi \\ cos\theta cos\phi \end{bmatrix} + \begin{bmatrix} X + T \\ Y \\ Z \end{bmatrix}$$

Moment Equations

$$\begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix}\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \left(I_y - I_z\right)qr + I_{xz}pq \\ \left(I_z - I_x\right)rq + I_{xz}\left(r^2 - p^2\right) \\ \left(I_x - I_y\right)pq + I_{xz}qr \end{bmatrix} + \begin{bmatrix} L \\ M + \ell_T T \\ N \end{bmatrix}$$

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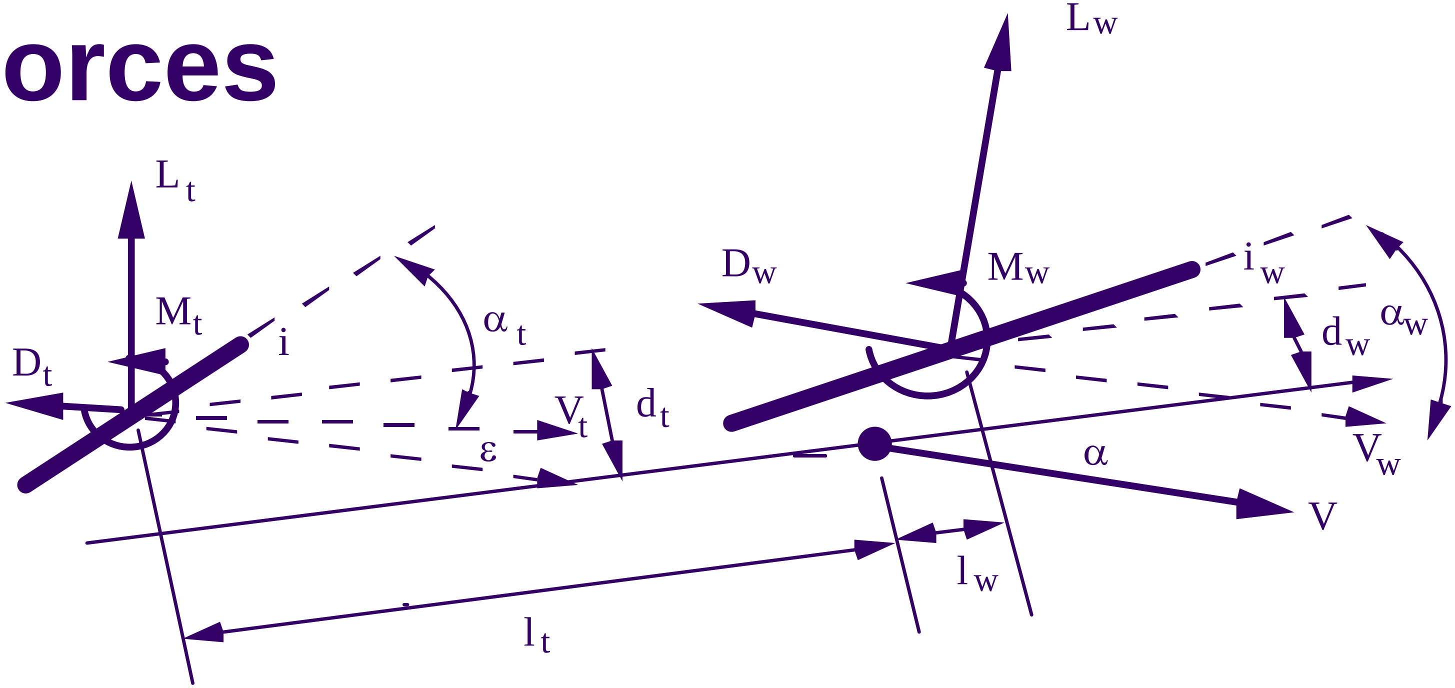
Longitudinal Aerodynamic Forces



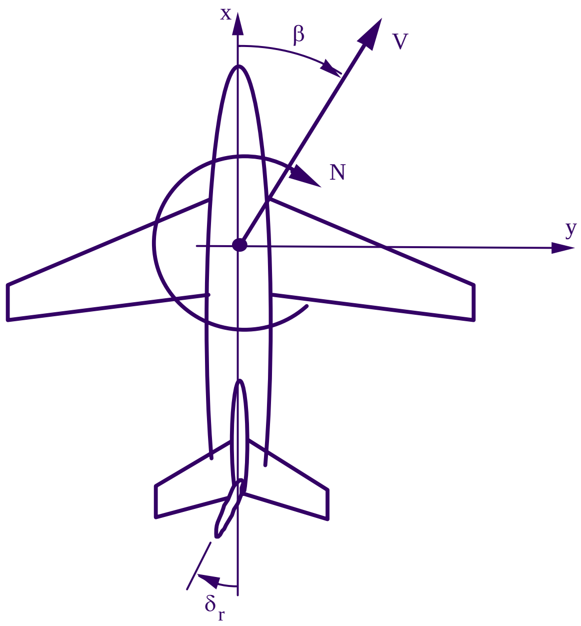
Wing angle of attack: $\alpha_w = \alpha + i_w$

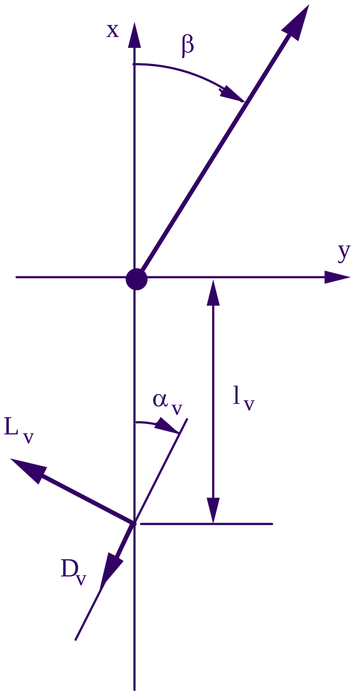
Tail angle of attack: $\alpha_t = \alpha + i_t - \varepsilon + \delta_e + \left(\ell_t / V\right)\dot{\theta}$

(ε is downwash angle, i_w, i_t are incidence angles of wing and tail)

Yaw Forces and Moments

The principle aerodynamic yaw forces are divide into two parts: (1) those due to wings and fuselage, and (2) those produced by the vertical tail





Wings & Fuselage

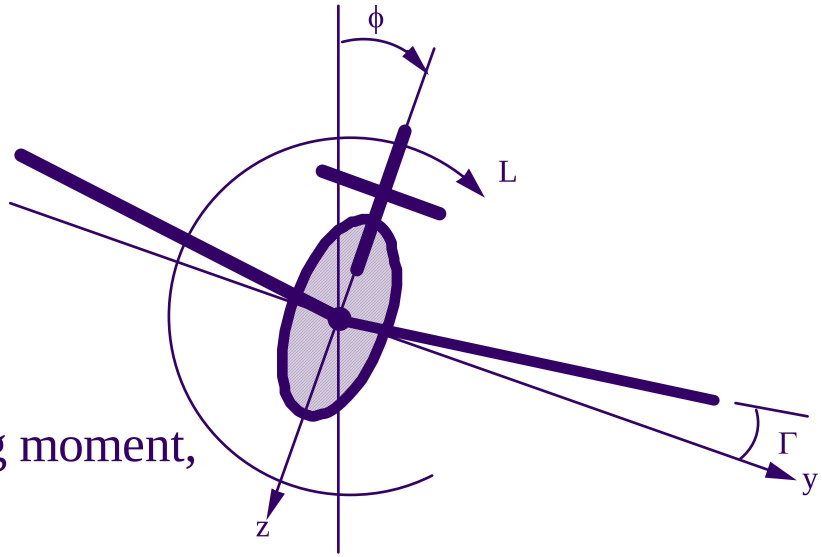
$$L_{wf} = C_{L_{wf}}(\beta)\tfrac{1}{2}\rho V^2 S_f \quad D_{wf} = C_{D_{wf}}(\beta)\tfrac{1}{2}\rho V^2 S_f$$

Vertical Tail

$$L_v = C_{L_v}(\alpha_v)\tfrac{1}{2}\rho V^2 S_v \quad D_v = C_{D_v}(\alpha_v)\tfrac{1}{2}\rho V^2 S_v \quad \alpha_v = \beta + \sigma - a_r\delta_r$$

$(\sigma,$ sidewash due to wing influence$)$

Roll Forces



Roll moment has four primary sources:

- vertical tail lift produces restoring moment
- fuselage, if wings are high produces restoring moment, low produces destabilizing moment
- wing dihedral, Γ, produces restoring moment (tail dihedral also, but often negligible)
- ailerons, δ_a

If positive wing dihedral is present, any sideslip velocity alters the local wing angle of attack increasing the angle of attack on the downwind wing and decreasing the angle of attack on the upwind wing causing both roll and yaw moments. For small dihedral angle , the change in andgle of attack is $\pm V \sin\beta\,\Gamma$.

A positive aileron angle δ_a changes the left wing angle of attack by an amount $\tau\delta_a$ and the right wing angle of attack by an amount $-\tau\delta_a$ thereby producing a positive roll moment.

Aerodynamic Forces: Nondimensional Coefficients

In general, the aerodynamic forces and moments are expressed in terms of aerodynamic coefficients:

- Forces:

  $X = C_X QS$ *(axial force)*

  $Y = C_Y QS$ *(side force)*

  $Z = C_Z QS$ *(normal force)*

- Moments:

  $L = C_L QSl$ *(rolling moment)*

  $M = C_M QSl$ *(pitching moment)*

  $N = C_N QSl$ *(yawing moment)*

$Q := \rho V^2/2$ *(dynamic pressure),*

$S :=$ *characteristic area (wing surface),* $l :=$ *characteristic length (b=wing span or c=mean chord)*

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Aerodynamic Forces: Example

$$C_X(\alpha,\delta_e) = C_x(\alpha,\delta_e)$$

$$C_Y(\alpha,\beta,\delta_r,\delta_a,p,r) = C_{y\beta}(\alpha)\beta + C_{y\delta_r}(\alpha)\delta_r + C_{y\delta_a}(\alpha)\delta_a + \frac{b}{2V}\left(C_{yp}(\alpha)p + C_{yr}(\alpha)r\right)$$

$$C_Z(\alpha,\delta_e) = C_z(\alpha,\delta_e)$$

$$C_L(\alpha,\beta,\delta_r,\delta_a,p,r) = C_{l\beta}(\alpha)\beta + C_{l\delta_r}(\alpha)\delta_r + C_{l\delta_a}(\alpha)\delta_a + \frac{b}{2V}\left(C_{lp}(\alpha)p + C_{lr}(\alpha)r\right)$$

$$C_M(\alpha,q,\delta_e) = C_m(\alpha,\delta_e) + \frac{c}{2V}C_{mq}(\alpha)q$$

$$C_N(\alpha,\beta,\delta_r,\delta_a,p,r) = C_{n\beta}(\alpha)\beta + C_{n\delta_r}(\alpha)\delta_r + C_{n\delta_a}(\alpha)\delta_a + \frac{b}{2V}\left(C_{np}(\alpha)p + C_{nr}(\alpha)r\right)$$

Notes: 1) In highly maneuverable fighters large sideslip velocities may impose nonlinear β dependence and β cross-coupling with longitudinal forces and moments. Also, large pitch rates may induce pitch rate affects in longitudinal forces. 2) For aircraft with large changes in Mach number, such as the aerospace plane, all coefficients may require Mach number dependence.

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