

Flight Dynamics & Control

Dynamics Near Equilibria

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Outline

- Program
- Straight & Level Flight –the primary steady-state
- Perturbation Equations for Straight & Level Flight
- Longitudinal & Lateral Dynamics
- Coordinated Turn
- Steady Sideslip

Program

- Determine trim conditions
 - Define and compute steady-state flight conditions
 - Straight and level flight (cruise, climb, descend)
 - Coordinated turn
 - Steady sideslip
- Determine perturbation models
 - Small perturbations from a trim condition
 - Examine dynamical behaviors

Reduction in Dimension

coordinates: $x_s, y_s, z_s, \phi, \theta, \psi$ velocities: $u, v, w, p, q, r \Rightarrow 12$ states

Suppose 1) the earth is flat and 2) we ignore variations of air density

\Rightarrow $\left\{ \begin{array}{l} \text{the dynamics are invariant w.r.t. location } (x_s, y_s, z_s) \\ \text{the dynamics are invariant w.r.t. (inertial) heading} \end{array} \right.$

Consequently, we could drop x_s, y_s, z_s, ψ and study an 8 state system

- Notes:

- These properties do not require linearity
- These omitted variables can always be included by adding the appropriate kinematic equations – we often include z_s and/or ψ



Reduced Equations

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = m \begin{bmatrix} rv - qw \\ pw - ru \\ qu - pv \end{bmatrix} + mg \begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix} + \begin{bmatrix} X + T \\ Y \\ Z \end{bmatrix}$$

$$\begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} (I_y - I_z)qr + I_{xz}pq \\ (I_z - I_x)rq + I_{xz}(r^2 - p^2) \\ (I_x - I_y)pq + I_{xz}qr \end{bmatrix} + \begin{bmatrix} L \\ M + \ell_T T \\ N \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Equilibria

The equations are organized in first order standard form, involving the state vector x , input vector u , and output vector y .

$$E(x)\dot{x} = f(x, u) \text{ state equations}$$

$$y = h(x, u) \text{ output equations}$$

A set of values (x_0, u_0, y_0) is called an equilibrium (or trim) point if it satisfies:

$$0 = f(x_0, u_0)$$

$$y_0 = h(x_0, u_0)$$

We are interested in motions that remain close to the equilibrium point.



The Primary Steady State: Straight & Level flight

The straight and level flight condition requires equilibrium flight along a linear path with constant flight path angle γ^* , constant velocity V^* , zero sideslip and wings level. Thus, we impose the following conditions for straight and level flight:

Equilibrium:

$$\dot{u} = 0, \dot{v} = 0, \dot{w} = 0 \left(\dot{V} = 0, \dot{\alpha} = 0, \dot{\beta} = 0 \right),$$

$$\dot{p} = 0, \dot{q} = 0, \dot{r} = 0, \dot{\phi} = 0, \dot{\theta} = 0, \dot{\psi} = 0 \left(\omega = 0 \right)$$

Outputs:

speed: $V = V^*$

flight path angle: $\gamma := \theta - \alpha = \gamma^*$

sideslip: $\beta = 0$

roll: $\phi = 0$



Straight & Level Flight

Proposition : An equilibrium point satisfying the straight and level flight conditions exists if and only if there exists $\alpha^*, \delta_e^*, T^*$ which satisfy the equations:

$$\begin{aligned} X(V^*, \alpha^*, \delta_e^*) - mg \sin(\alpha^* + \gamma^*) + T^* &= 0, & Z(V^*, \alpha^*, \delta_e^*) + mg \cos(\alpha^* + \gamma^*) &= 0, \\ M(V^*, \alpha^*, \delta_e^*, 0) + \ell_T T^* &= 0 \end{aligned}$$

In this case the equilibrium values of the states and controls are:

longitudinal variables;

states: $V = V^*, \alpha = \alpha^*, q = 0, \theta = \gamma^* + \alpha^*$ controls: $\delta_e = \delta_e^*, T = T^*$

lateral variables;

states: $\beta = 0, p = 0, r = 0, \phi = 0$, controls: $\delta_r = 0, \delta_a = 0$



Linearization

Define: $x(t) = x_0 + \delta x(t)$, $u(t) = u_0 + \delta u(t)$, $y(t) = y_0 + \delta y(t)$

The equations become: $E(x_0)\delta\dot{x} = f(x_0 + \delta x(t), u_0 + \delta u(t))$

$$y_0 + \delta y(t) = h(x_0 + \delta x(t), u_0 + \delta u(t))$$

Now, construct a Taylor series for f, h

$$f(x_0 + \delta x, u_0 + \delta u) = f(x_0, u_0) + \frac{\partial f(x_0, u_0)}{\partial x} \delta x + \frac{\partial f(x_0, u_0)}{\partial u} \delta u + \text{hot}$$

$$h(x_0 + \delta x, u_0 + \delta u) = h(x_0, u_0) + \frac{\partial h(x_0, u_0)}{\partial x} \delta x + \frac{\partial h(x_0, u_0)}{\partial u} \delta u + \text{hot}$$

Notice that $f(x_0, u_0) = 0$ and $h(x_0, u_0) = y_0$, so

$$E(x_0)\delta\dot{x} = \frac{\partial f(x_0, u_0)}{\partial x} \delta x + \frac{\partial f(x_0, u_0)}{\partial u} \delta u$$

$$\delta y = \frac{\partial h(x_0, u_0)}{\partial x} \delta x + \frac{\partial h(x_0, u_0)}{\partial u} \delta u$$

\Rightarrow

$$\boxed{\begin{aligned} E\delta\dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u \end{aligned}}$$

Perturbations for Straight & Level Flight ~ longitudinal equations

$$\begin{bmatrix} m \cos \alpha & -mV \sin \alpha & mV \sin \alpha & 0 \\ m \sin \alpha & mV \cos \alpha & -mV \cos \alpha & 0 \\ 0 & 0 & I_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \Delta V \\ \Delta \alpha \\ q \\ \Delta \theta \end{bmatrix} =$$

$$\begin{bmatrix} \frac{2}{V} QSC_x(\alpha, \delta_e) & QSC_{x\alpha}(\alpha, \delta_e) & 0 & -mg \cos \theta \\ \frac{2}{V} QSC_z(\alpha, \delta_e) & QSC_{z\alpha}(\alpha, \delta_e) & V \cos \alpha & 0 \\ \frac{2}{V} QSlC_M(\alpha, \delta_e) & QSlC_{m\alpha}(\alpha, \delta_e) & \frac{2}{V} QSlC_{mq}(\alpha, \delta_e) & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \alpha \\ q \\ \Delta \theta \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & QSC_{x\delta_e}(\alpha, \delta_e) \\ 0 & QSC_{z\delta_e}(\alpha, \delta_e) \\ l_T & QSC_{m\delta_e}(\alpha, \delta_e) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta T \\ \Delta \delta_e \end{bmatrix}$$

Perturbations for Straight & Level Flight ~ lateral equations

$$\begin{bmatrix} mV & 0 & 0 & 0 \\ 0 & I_x & -I_{xy} & 0 \\ 0 & -I_{xy} & I_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} =$$

$$\begin{bmatrix} QSC_{y\beta}(\alpha) & QS \frac{b}{2V} C_{y\beta}(\alpha) & QS \frac{b}{2V} C_{yr}(\alpha) - mu & -mg \cos \theta \\ QS \ell C_{l\beta}(\alpha) & QS \ell \frac{b}{2V} C_{lp}(\alpha) & QS \ell \frac{b}{2V} C_{lr}(\alpha) & 0 \\ QS \ell C_{n\beta}(\alpha) & QS \ell \frac{b}{2V} C_{np}(\alpha) & QS \ell \frac{b}{2V} C_{nr}(\alpha) & 0 \\ 0 & 1 & \tan \theta & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix}$$

$$+ QS \begin{bmatrix} C_{y\delta_r}(\alpha) & C_{y\delta_a}(\alpha) \\ \ell C_{l\delta_r}(\alpha) & \ell C_{l\delta_a}(\alpha) \\ \ell C_{n\delta_r}(\alpha) & \ell C_{n\delta_a}(\alpha) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix}$$

Longitudinal Dynamics

Near straight and level flight the general equations of motion may be divided into two sets which are almost decoupled.

Longitudinal Dynamics : Motion in body x-z plane, without yawing or rolling ($v = 0, \phi = 0, \psi = 0, p = 0, r = 0$).

Longitudinal Variables : $u, w, q, \theta, x_s, y_s$

Uncoupled longitudinal motions exist provided $\left\{ \begin{array}{l} \text{the } x - z \text{ plane is a plane of symmetry} \\ \text{rotor gyroscopic effects are absent} \\ \text{the flat earth approximation is valid} \end{array} \right.$

$$\text{Force equations: } \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = m \begin{bmatrix} qw \\ qu \end{bmatrix} + mg \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} + \begin{bmatrix} X \\ Z \end{bmatrix}$$

$$\text{Moment equations: } I_y \dot{q} = M$$

$$\text{Kinematics: } \begin{bmatrix} \dot{x}_s \\ \dot{z}_s \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad \dot{\theta} = q$$



Lateral Dynamics

Lateral Dynamics: motion in body x-y plane, no pitching

$$(w \approx 0, u \approx u^*, \theta = \theta^*, q = 0, \alpha \approx \alpha^*)$$

Lateral Variables: v, p, r, ψ, ϕ, y_s

Uncoupled lateral motions exist provided

- motions are small-trajectory remains close to straight & level
- aerodynamic cross-coupling terms are negligible
- rotor gyroscopic effects are absent
- the flat earth approximation is valid

Force equations: $m\dot{v} = -mru^* + mg \cos \theta^* \sin \phi + Y$

Moment equations:
$$\begin{bmatrix} I_x & -I_{xz} \\ -I_{xz} & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L \\ N \end{bmatrix}$$

Kinematics:
$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \cos \phi \tan \theta^* \\ & \cos \phi \sec \theta^* \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix}$$

$$\dot{y}_s = \cos \theta^* \sin \phi u^* + (\sin \phi \sin \theta^* \sin \psi + \cos \phi \cos \psi) v$$



Banked Coordinated Turn

The banked turn is defined by the following conditions:

1) equilibrium - $V, \alpha, \beta(u, v, w), p, q, r$ are constant

$$0 = m \begin{bmatrix} rv - qw \\ pw - ru \\ qu - pv \end{bmatrix} + mg \begin{bmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{bmatrix} + \begin{bmatrix} X + T \\ Y \\ Z \end{bmatrix}$$

$$0 = \begin{bmatrix} (I_y - I_z)qr + I_{xz}pq \\ (I_z - I_x)rq + I_{xz}(r^2 - p^2) \\ (I_x - I_y)pq + I_{xz}qr \end{bmatrix} + \begin{bmatrix} L \\ M + \ell_T T \\ N \end{bmatrix}$$

Banked Coordinated Turn~2

2) banked turn condition:

the (inertial) angular velocity is vertical and constant

$$\omega_s = \begin{bmatrix} 0 \\ 0 \\ \omega^* \end{bmatrix} \Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{bmatrix} \omega^*$$

3) coordinated turn condition - sum of gravity and inertial forces lie in plane of symmetry ($x - z$ plane)

$$mpw - mru + mg \cos \theta \sin \phi = 0 \Rightarrow$$

$$pV \cos \beta \sin \alpha - rV \cos \beta \cos \alpha + g \cos \theta \sin \phi = 0$$

4) climb conditions: $V = V^*$, $\gamma = \gamma^*$



Banked Coordinated Turn ~ 3

- There are 12 equations in 12 unknowns

$$V, \alpha, \beta, p, q, r, \theta, \phi, T, \delta_e, \delta_a, \delta_r$$

- The fact that the velocity is constant in body frame, with constant angular velocity about z_s insures that ground track is circular.
- The coordinated turn condition insures that pilot and passengers will not experience any side forces.
- A pilot achieves coordination by using the rudder in conjunction with an instrument called a turn coordinator which measures the difference between the inertial and gravity forces acting along the y axis.
- The rudder also induces a moment which counteracts the “adverse yaw” moment resulting from increased (decreased) drag on the outside (inside) wing produced by aileron position and which can be significant during the rolling phase of the turn.

Banked Coordinated Turn ~ 4

5 equations explicitly yield 5 unknowns (p, q, r, V, α or θ) in terms of the others, so these can be eliminated, leaving:

$$pV \cos \beta \sin \alpha - rV \cos \beta \cos \alpha + g \cos \theta \sin \phi = 0$$

$$m \begin{bmatrix} rv - qw \\ 0 \\ qu - pv \end{bmatrix} + mg \begin{bmatrix} -\sin \theta \\ 0 \\ \cos \theta \cos \phi \end{bmatrix} + \begin{bmatrix} X + T \\ Y \\ Z \end{bmatrix} = 0$$

$$\begin{bmatrix} (I_y - I_z)qr + I_{xz}pq \\ (I_z - I_x)rq + I_{xz}(r^2 - p^2) \\ (I_x - I_y)pq + I_{xz}qr \end{bmatrix} + \begin{bmatrix} L \\ M + \ell_T T \\ N \end{bmatrix} = 0$$

with p, q, r known functions of ω^*



Banked Coordinated Turn ~ 5

Consider the possibility of solutions in which ω is small (R large).

Proposition : There exists a solution corresponding to $\omega = 0$ if and only if there exist $\alpha^*, \delta_e^*, T^*$ satisfying

$$\begin{aligned} X(V^*, \alpha^*, \delta_e^*) - mg \sin(\alpha^* + \gamma^*) + T^* &= 0, & Z(V^*, \alpha^*, \delta_e^*) + mg \cos(\alpha^* + \gamma^*) &= 0, \\ M(V^*, \alpha^*, \delta_e^*, 0) + \ell_T T^* &= 0 \end{aligned}$$

This is the case of straight and level flight in which the equilibrium values of the states and controls are:

In this case the equilibrium values of the states and controls are:

longitudinal variables;

$$\text{states: } V = V^*, \alpha = \alpha^*, q = 0, \theta = \gamma^* + \alpha^* \quad \text{controls: } \delta_e = \delta_e^*, T = T^*$$

lateral variables;

$$\text{states: } \beta = 0, p = 0, r = 0, \phi = 0, \quad \text{controls: } \delta_r = 0, \delta_a = 0$$



Banked Coordinated Turn ~ 6

Proposition : Suppose $\alpha^*, \delta_e^*, T^*$ constitute a regular solution of the longitudinal equilibrium equations and

$$\det \begin{bmatrix} C_{y\beta}(\alpha) & C_{y\delta_r}(\alpha) & 0 \\ C_{l\beta}(\alpha) & C_{l\delta_r}(\alpha) & C_{l\delta_a}(\alpha) \\ C_{n\beta}(\alpha) & C_{n\beta}(\alpha) & C_{n\beta}(\alpha) \end{bmatrix} \neq 0$$

Then there exists a solution to the coordinated bank turn equations, with $\beta, \delta_r, \delta_a$ small and α, δ_e, T near $\alpha^*, \delta_e^*, T^*$. To first order terms, the solution is

$$\tan \phi = \frac{\omega V^* \cos \gamma^*}{g}$$

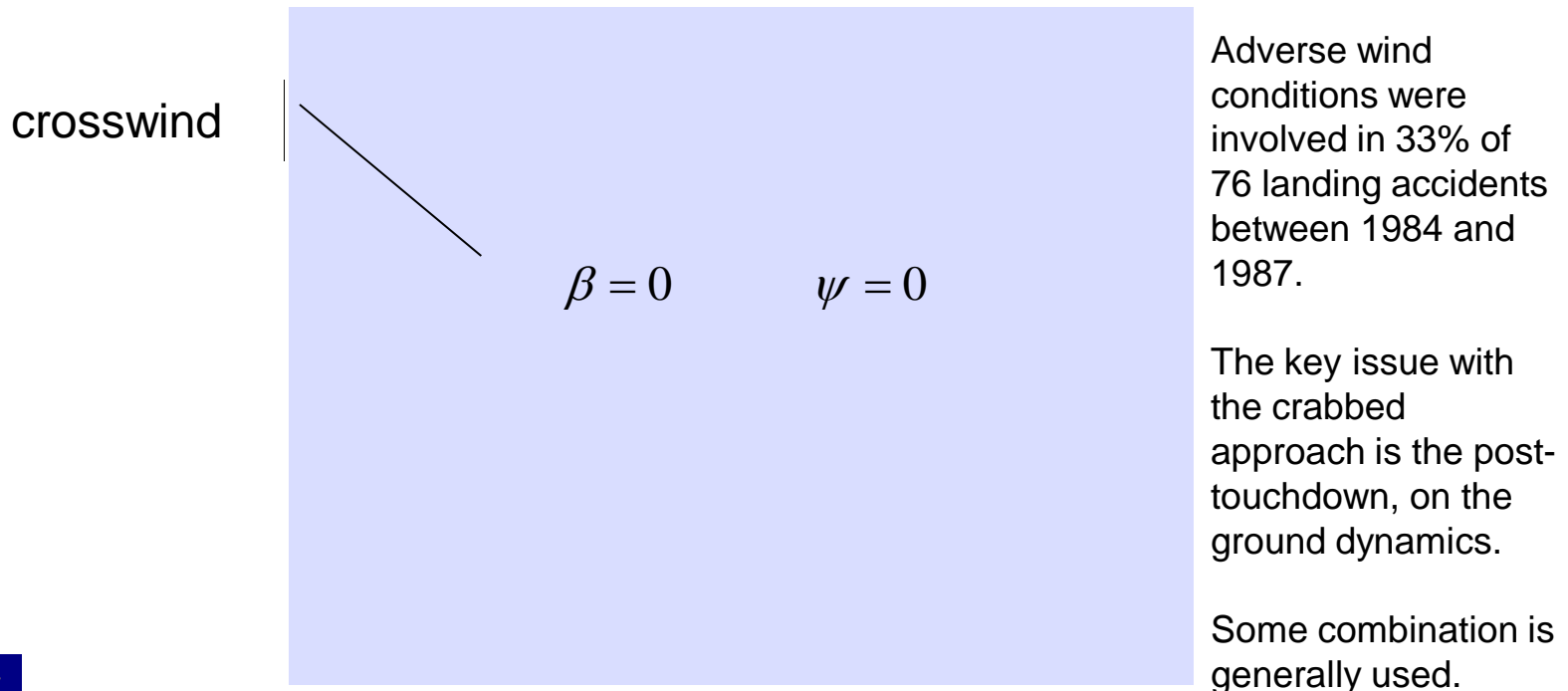
$$\begin{bmatrix} C_{y\beta}(\alpha) & C_{y\delta_r}(\alpha) & 0 \\ C_{l\beta}(\alpha) & C_{l\delta_r}(\alpha) & C_{l\delta_a}(\alpha) \\ C_{n\beta}(\alpha) & C_{n\beta}(\alpha) & C_{n\beta}(\alpha) \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} C_{yp}(\alpha) & C_{yr}(\alpha) \\ C_{lp}(\alpha) & C_{lr}(\alpha) \\ C_{np}(\alpha) & C_{nr}(\alpha) \end{bmatrix} \begin{bmatrix} 0 \\ -\cos \phi \end{bmatrix} \frac{\omega b}{2V^*}$$

$$\begin{bmatrix} C_{m\alpha} & C_{m\delta_e} \\ C_{l\alpha} & C_{l\delta_e} \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta\delta_e \end{bmatrix} = - \begin{bmatrix} C_{mq} \\ C_{lq} \end{bmatrix} \frac{\omega c}{2V^*} \sin \phi + \begin{bmatrix} 0 \\ (\sec \phi - 1) C_{lw} \end{bmatrix}$$



Crosswind

The steady sideslip is a flight condition that may be used during landing approaches in the presence of a crosswind. It is an equilibrium condition in the sense that all accelerations and angular rates are zero. In addition, the aircraft tracks a linear ground path corresponding to $y_s=0$, i.e., aligned with the x_s axis. Flight path angle and speed are also specified.



Crosswind ~ 2

Consider an inertial (flat earth fixed) frame with $x_s - y_s$ coordinates on the surface and z_s down. The intent is to land along the x_s -axis (so we require $y_s = 0$). Assume the crosswind velocity is v_{s0} . Thus, we have the following conditions:

Regulated outputs:

$$y_s = 0 \Rightarrow$$

$$\dot{y}_s = v_{s0} + V \cos \beta \cos \alpha \cos \theta \sin \psi + V \sin \beta (\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi) \\ + V \cos \beta \sin \alpha (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) = 0$$

$$V = V^*$$

$$\gamma := \theta - \alpha = \gamma^*$$

Equilibrium conditions:

$$\dot{V} = 0, \dot{\alpha} = 0, \dot{\beta} = 0, \dot{q} = 0, \dot{\theta} = 0, \dot{p} = 0, \dot{r} = 0, \dot{\phi} = 0, \dot{\psi} = 0$$



Crosswind ~ 3

Proposition : (crabbing solutions) A solution to the ground tracking problem exists with $\beta = 0, \phi = 0$ if and only if there exist α^*, δ_e^* and T^* which satisfy:

$$\begin{aligned} X(V^*, \alpha^*, \delta_e^*) - mg \sin(\alpha^* + \gamma^*) + T^* &= 0, & Z(V^*, \alpha^*, \delta_e^*) + mg \cos(\alpha^* + \gamma^*) &= 0, \\ M(V^*, \alpha^*, \delta_e^*, 0) + \ell_T T^* &= 0 \end{aligned}$$

and

$$\left| \frac{v_{s0}}{V^* \cos \gamma^*} \right| \leq 1$$

Crosswind ~ 4

Proposition : (steady sideslip) A solution to the ground tracking problem exists with $\psi = 0$ if and only if there exist α^* , δ_e^* and T^* which satisfy:

$$\begin{aligned} X(V^*, \alpha^*, \delta_e^*) - mg \sin(\alpha^* + \gamma^*) + T^* &= 0, & Z(V^*, \alpha^*, \delta_e^*) + mg \cos(\alpha^* + \gamma^*) &= 0, \\ M(V^*, \alpha^*, \delta_e^*, 0) + \ell_T T^* &= 0 \end{aligned}$$

and ϕ^* , β^* , δ_r^* , δ_a^* satisfying

$$\frac{mg \cos \theta \sin \phi}{QS\ell} + C_{y\beta}(\alpha)\beta + C_{y\delta_r}(\alpha)\delta_r + C_{y\delta_a}(\alpha)\delta_a = 0$$

$$C_{l\beta}(\alpha)\beta + C_{l\delta_r}(\alpha)\delta_r + C_{l\delta_a}(\alpha)\delta_a = 0$$

$$C_{n\beta}(\alpha)\beta + C_{n\delta_r}(\alpha)\delta_r + C_{n\delta_a}(\alpha)\delta_a = 0$$

$$\frac{v_{s0}}{V} + \cos \phi \sin \beta = 0$$

Crosswind Example

A hypothetical subsonic transport (adapted from Etkin) has the following data

$$C_{y\beta} = -0.168, C_{y\delta_r} = 0.067, C_{y\delta_a} = 0,$$

$$C_{l\beta} = -0.047, C_{l\delta_r} = 0.003, C_{l\delta_a} = -0.04,$$

$$C_{n\beta} = 0.3625, C_{n\delta_r} = -0.16, C_{n\delta_a} = -0.005$$

Suppose the crosswind is $v_0 = 0.15V$. Then the following results are obtained (in degrees)

$$\beta = -8.59437, \phi = -0.121212, \delta_r = -19.7409, \delta_a = 8.61781$$

Slip to the left, roll to the left, hard right rudder, left aileron (into wind)



Comments from Airbus Pilots

- “I do not find crosswinds to be anymore challenging in this airplane than any other. You have to understand that you cannot "slip" this airplane because you are commanding a ROLL RATE with the Side Stick Controller, not a BANK ANGLE. Here is a suggestion: Allow the airplane to do an Auto Land in a crosswind when it is convenient, and VFR. You will be shocked at the timing of when the airplane leaves the CRAB and applies rudder to align the nose parallel with the runway. You think it just isn't going to do it, and at the very last second, it slides it in perfectly. I would guess in the last 20 feet or less. My technique is just the same as any airplane I've ever flown. C-150 to B-767. Crab it down to the flare, apply enough rudder to straighten the nose, drop the up-wind wing to prevent drift, land on the up-wind main first.” -- 5-Year Airbus Captain.