Dynamical Systems & Lyapunov Stability

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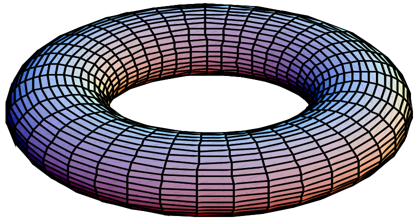
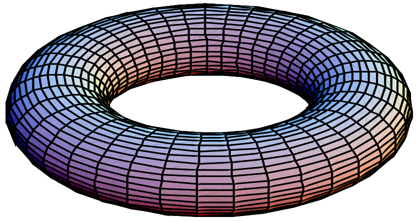


Outline

- Ordinary Differential Equations
  - Existence & uniqueness
  - Continuous dependence on parameters
  - Invariant sets, nonwandering sets, limit sets
- Lyapunov Stability
  - Autonomous systems
  - Basic stability theorems
  - Stable, unstable & center manifolds
  - Control Lyapunov function

Basics of Nonlinear ODE's



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Dynamical Systems

$$\frac{d}{dt}x(t) = f\left(x(t),t\right), \quad x \in R^n, t \in R \quad \text{non-autonomous}$$

$$\frac{d}{dt}x(t) = f\left(x(t)\right), \quad x \in R^n, t \in R \quad \text{autonomous}$$

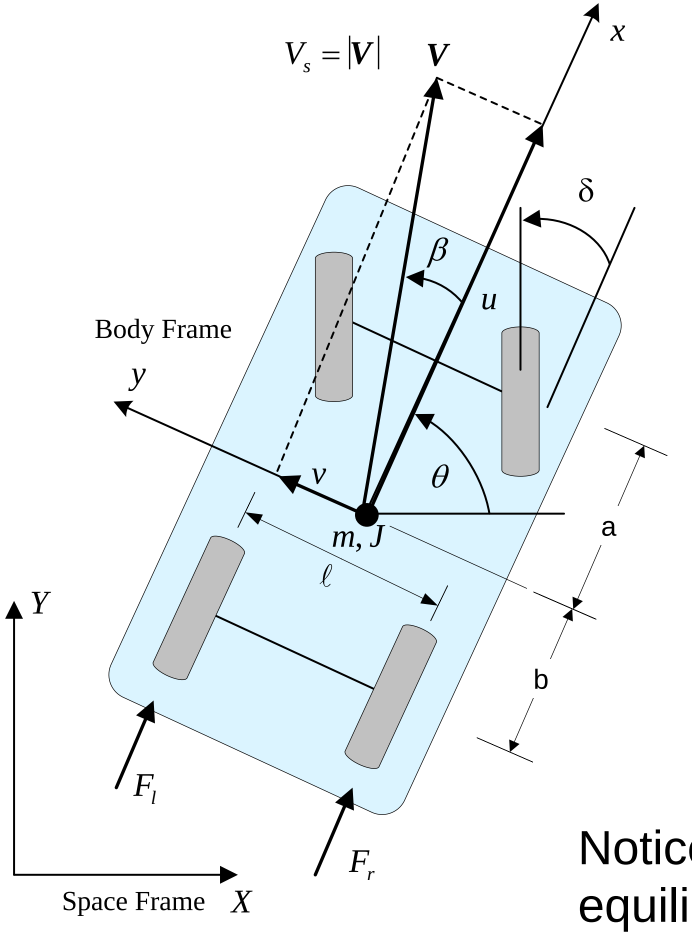
A solution on a time interval $t \in \left[t_0, t_1\right]$ is a function

$x(t):[t_0,\ t_1] \to R^n$ that satisfies the ode.

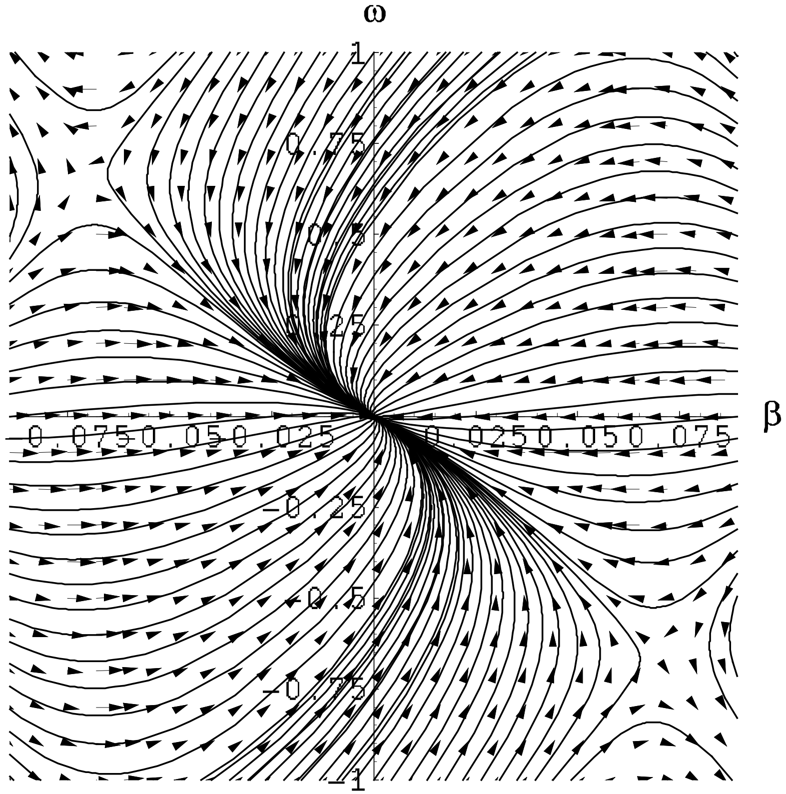
Vector Fields and Flow

- We can visualize an individual solution as a graph $x(t): t \to R^n$.
- For autonomous systems it is convenient to think of $f(x)$ as a vector field on R^n - $f(x)$ assigns a vector to each point in R^n. As t varies, a solution $x(t)$ traces a path through R^n tangent to the field $f(x)$.
- These curves are often called trajectories or orbits.
- The collection of all trajectories in R^n is called the flow of the vector field $f(x)$.

Auto at Constant Speed



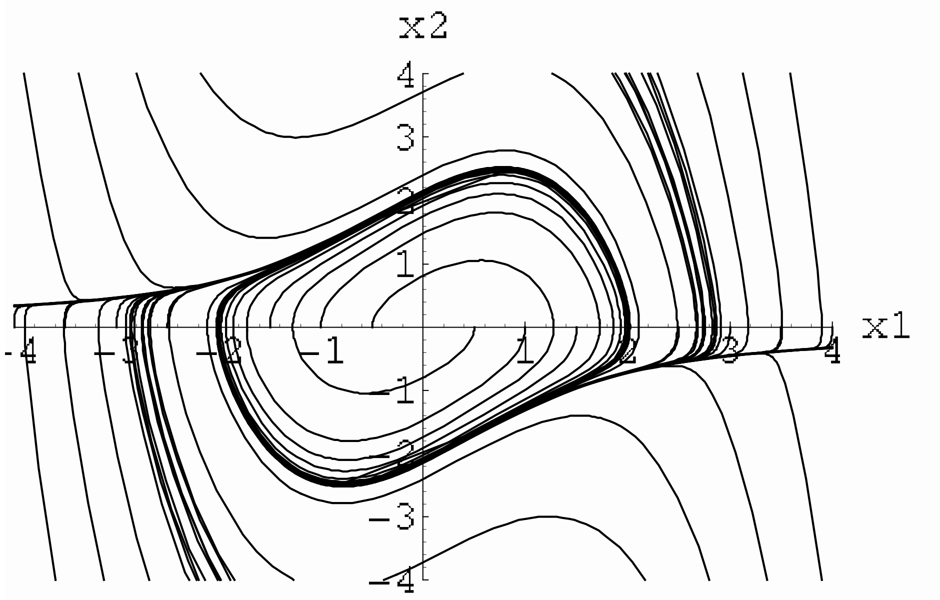
$$\frac{d}{dt}\begin{bmatrix}\omega \\ \beta\end{bmatrix} = f\left(\omega, \beta, \bar{V}, \delta\right)$$



Notice the three equilibria.

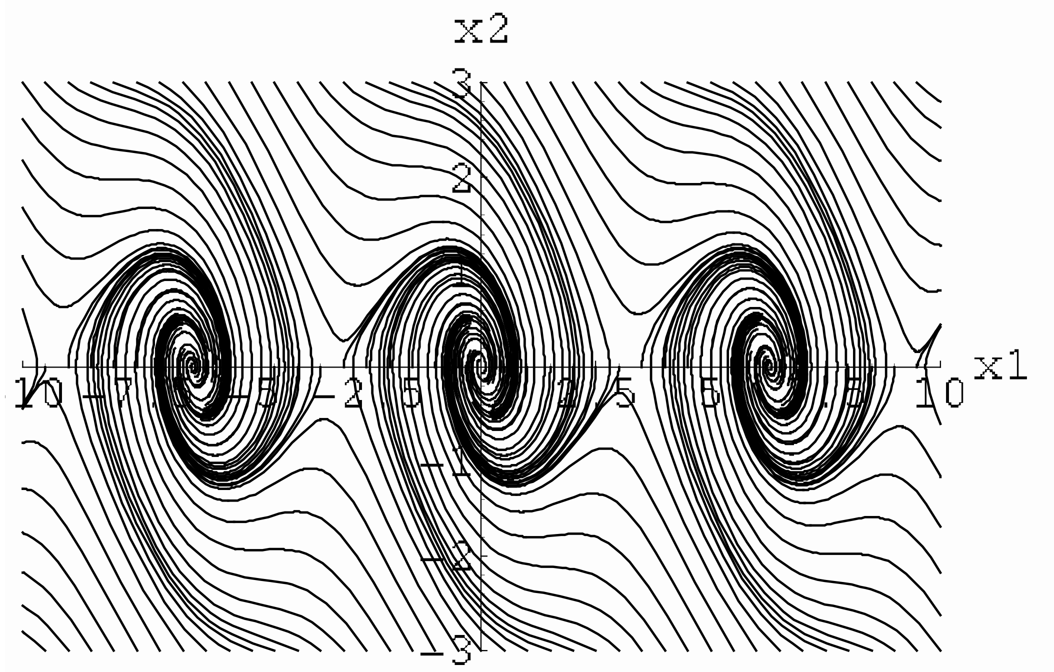
Van der Pol

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.8\left(1 - x_1^2\right)x_2 - x_1 \end{bmatrix}$$



Damped Pendulum

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2/2 - \sin x_1 \end{bmatrix}$$



Lipschitz Condition

The existence and uniqueness of solutions depend on properties of the function f. In many applications $f(x,t)$ has continuous derivatives in *x*. We relax this - we require that f is **Lipschitz** in *x*.

Def : $f: R^n \rightarrow R^n$ is locally Lipschitz on an open subset $D \subset R^n$ if each point $x_0 \in D$ has a neighborhood U_0 such that

$$\left\| f(x) - f(x_0) \right\| \leq L \left\| x - x_0 \right\|$$

for some constant L and all $x \in U_0$

Note: C^0 (continuous) functions need not be Lipschitz, C^1 functions always are.

The Lipschitz Condition

- A Lipschitz continuous function is limited in how fast it can change,
- A line joining any two points on the graph of this function will never have a slope steeper than its Lipschitz constant L,
- The mean value theorem can be used to prove that any differentiable function with bounded derivative is Lipschitz continuous, with the Lipschitz constant being the largest magnitude of the derivative.

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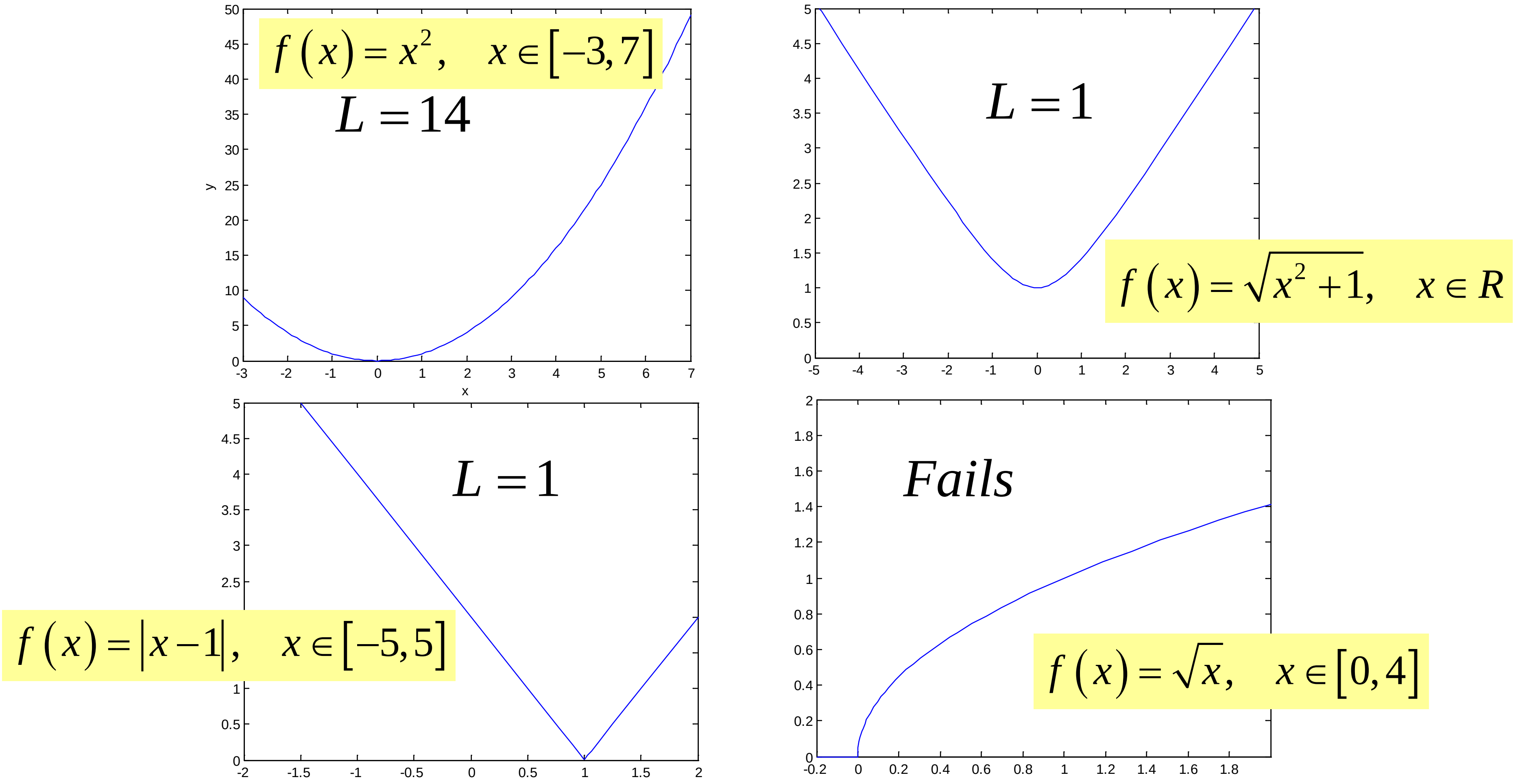
Examples: Lipschitz



Local Existence & Uniqueness

Proposition (Local Existence and Uniqueness) Let $f(x,t)$ be piece-wise continuous in t and satisfy the Lipschitz condition

$$\left\| f(x,t) - f(y,t) \right\| \leq L \left\| x - y \right\|$$

for all $x, y \in B_r = \left\{ x \in R^n \,\middle|\, \left\| x - x_0 \right\| < r \right\}$ and all $t \in [t_0, t_1]$. Then there exists $\delta > 0$ such that the differential equation with initial condition

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \in B_r$$

has a unique solution over $[t_0, t_0 + \delta]$.

The Flow of a Vector Field

$\dot{x} = f(x), \quad x(t_0) = x_0 \Rightarrow x(x_0, t)$ this notation indicates 'the solution of the ode that passes through x_0 at $t = 0$'

More generally, let $\Psi(x,t)$ denote the solution that passes through x at $t = 0$. The function $\Psi : R^n \times R \to R^n$ satisfies

$$\boxed{\frac{\partial \Psi(x,t)}{\partial t} = f(\Psi(x,t)), \quad \Psi(x,0) = x}$$

Ψ is called the flow or flow function of the vector field f

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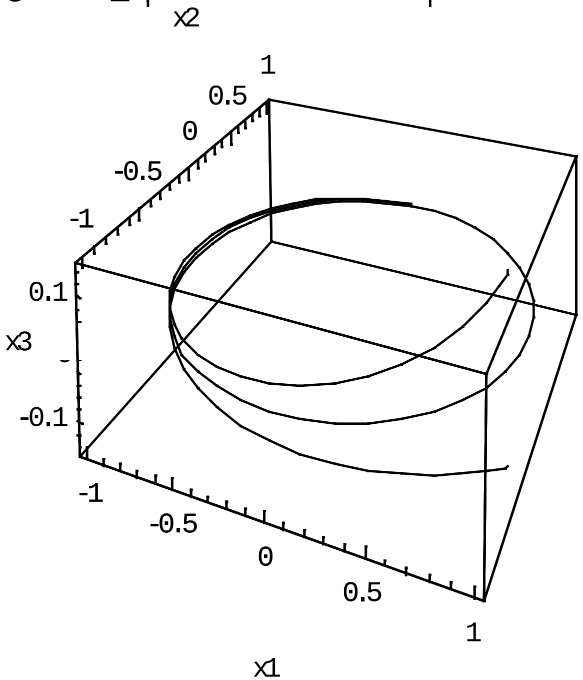
Example: Flow of a Linear Vector Field

$$\dot{x} = Ax \Rightarrow \frac{\partial \Psi(x,t)}{\partial t} = A\Psi(x,t) \Rightarrow \boxed{\Psi(x,t) = e^{At}x}$$

Example:

$$x \in R^3, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \Psi(x,t) = \begin{bmatrix} x_1\cos(t) + x_2\sin(t) \\ x_2\cos(t) - x_1\sin(t) \\ e^{-t}x_3 \end{bmatrix}$$



Invariant Set

A set of points $S \subset R^n$ is invariant with respect to the vector field f if trajectories beginning in S remain in S both forward and backward in time.

Examples of invariant sets:

any entire trajectory (equilibrium points, limit cycles)

collections of entire trajectories

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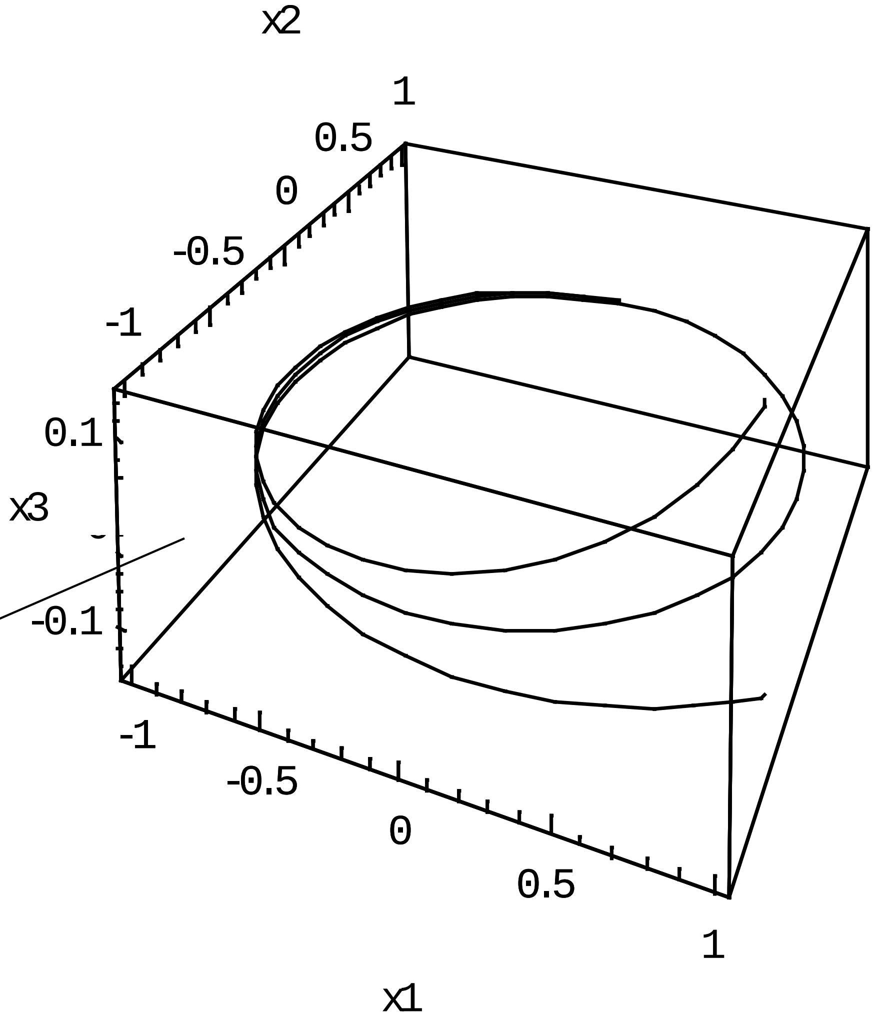
Example: Invariant Set

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



- each of the three trajectories shown are invariant sets
- the x_1-x_2 plane is an invariant set

Limit Points & Sets

A point $q \in R^n$ is called an ω-limit point of the trajectory $\Psi(t, p)$ if there exists a sequence of time values $t_k \to +\infty$ such that

$$\lim_{t_k \to \infty} \Psi(t_k, p) = q$$

q is said to be an α-limit point of $\Psi(t, p)$ if there exists a sequence of time values $t_k \to -\infty$ such that

$$\lim_{t_k \to -\infty} \Psi(t_k, p) = q$$

The set of all ω-limit points of the trajectory through p is the ω-limit set, and the set of all α-limit points is the α-limit set.

For an example see invariant set

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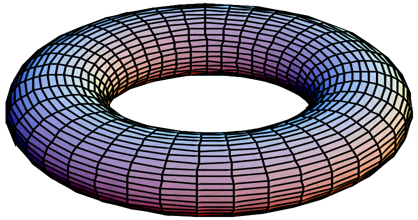
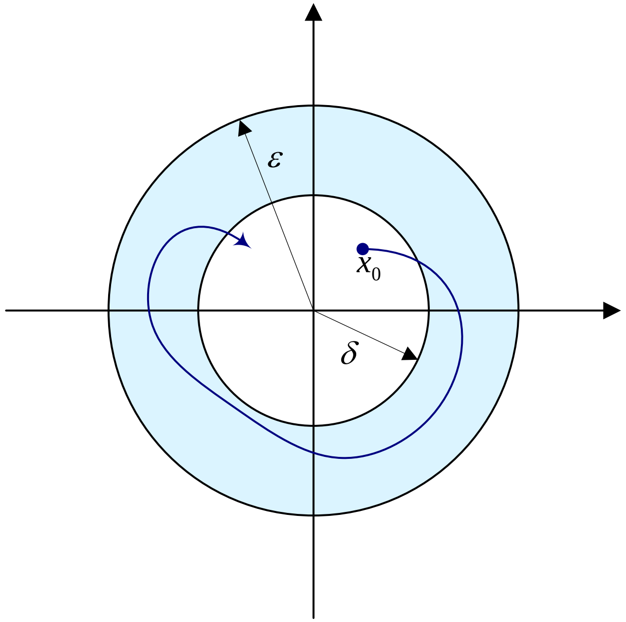
Introduction to Lyapunov Stability Analysis



Lyapunov Stability



$$\dot{x} = f(x), \quad f(0) = 0,$$

$$f : D \rightarrow R^n \ (locally\ Lipschitz)$$

The origin is

- a stable equilibrium point if for each $\varepsilon > 0,$ there is a $\delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \ \forall t > 0$$

- unstable if it is not stable, and
- asymptotically stable if δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) \rightarrow 0$$

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Two Simple Results

The origin is asymptotically stable only if it is isolated.

The origin of a linear system

$$\dot{x} = Ax$$

is stable if and only if $\left\| e^{At} \right\| \leq N < \infty \; \forall t > 0$

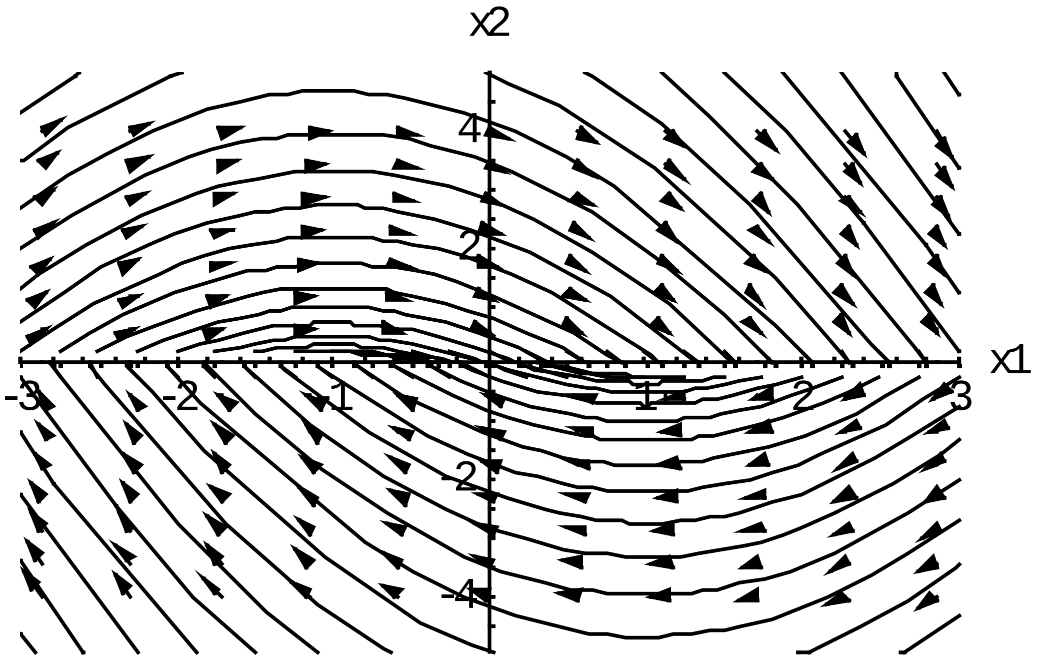
It is asymptotically stable if and only if, in addition

$$\left\| e^{At} \right\| \to 0, \; t \to \infty$$

Example: Non-isolated Equilibria

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -|x_2| x_1 - x_2 \end{bmatrix}$$

All points on the x_1 axis are equilibrium points



Positive Definite Functions

A function $V: R^n \to R$ is said to be

- positive definite if $V(0) = 0$ and $V(x) > 0,\ x \neq 0$
- positive semi-definite if $V(0) = 0$ and $V(x) \geq 0,\ x \neq 0$
- negative (semi-) definite if $-V(x)$ is positive (semi-) definite
- radially unbounded if $V(x) \to \infty$ as $\|x\| \to \infty$

For a quadratic form: $V(x) = x^T Q x,\ Q = Q^T$ the following are equivalent

- $V(x)$ is positive definite
- the eigenvalues of Q are positive
- the principal minors of Q are positive

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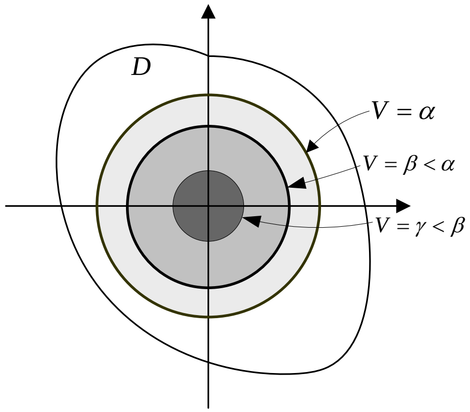
Lyapunov Stability Theorem

$V(x)$ is called a Lyapunov function relative to the flow of $\dot{x} = f(x)$ if it is positive definite and nonincreasing with respect to the flow:



$$V(0) = 0, \quad V(x) > 0 \ \ for\ x \neq 0$$

$$\dot{V} = \frac{\partial V(x)}{\partial x} f(x) \le 0$$

Theorem : If there exists a Lyapunov function on some neighborhood D of the origin, then the origin is stable. If $\dot{V}$ is negative definite on D then it is asymptotically stable.

Example: Linear System

$$\dot{x} = Ax$$

$$V(x) = x^T Q x, \quad Q^T = Q > 0$$

$$\dot{V} = x^T Q \dot{x} + \dot{x}^T Q x = x^T \left(QA + A^T Q \right) x = -x^T P x$$

where $QA + A^T Q = -P$ ← Lyapunov Equation

$P > 0 \Rightarrow A$ is stable (Hurwitz) : $\left\| e^{At} \right\|$ bounded $\forall t > 0$

So we can specify Q, compute P and test P.

Or, specify P and solve Lyapunov equation for Q and test Q.

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Example: Rotating Rigid Body

x, y, z body axes; $\omega_x, \omega_y, \omega_z$ angular velocities in body coord's;

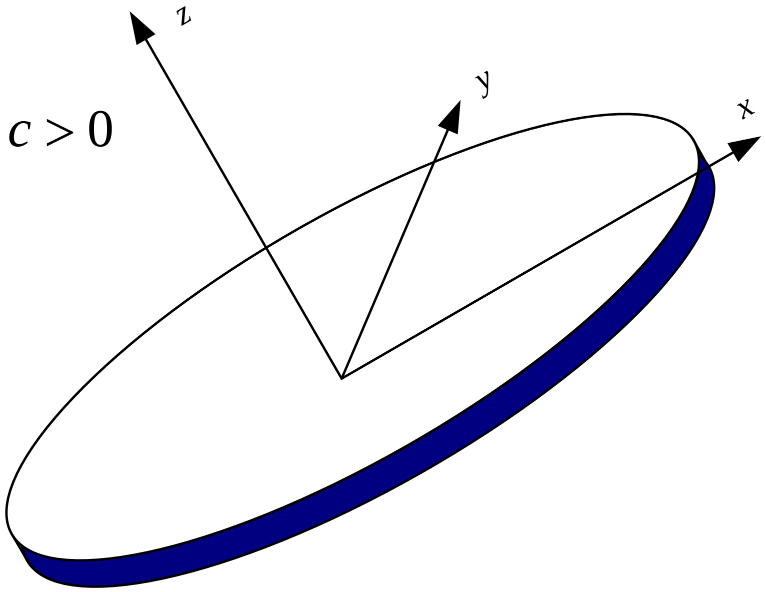
$\mathrm{diag}\left(I_x, I_y, I_z\right)$ $I_x \geq I_y \geq I_z > 0$ inertia matrix

$$\dot{\omega}_x = -\left(\frac{I_z - I_y}{I_x}\right)\omega_z\omega_y = a\omega_z\omega_y$$

$$\dot{\omega}_y = -\left(\frac{I_x - I_z}{I_y}\right)\omega_x\omega_z = -b\omega_x\omega_z$$

note $a, b, c > 0$

$$\dot{\omega}_z = -\left(\frac{I_y - I_x}{I_z}\right)\omega_y\omega_x = c\omega_y\omega_x$$



Rigid Body, Cont'd

Equilibrium requires:

$$0 = a\omega_z\omega_y$$

$$0 = -b\omega_x\omega_z \qquad a,b,c>0 \quad \Rightarrow$$

$$0 = c\omega_y\omega_x$$

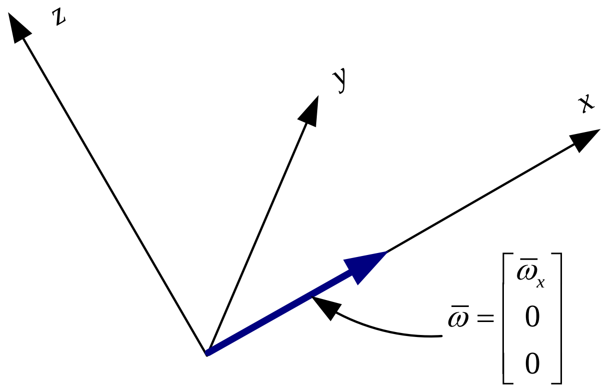
A state $(\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z)$ is an equilibrium point if any two of the angular velocity components are zero, i.e., the $\omega_x, \omega_y, \omega_z$ axes are all equilibrium points.

Consider a point $(\bar{\omega}_x, 0, 0)$. Shift $\omega_x \to \omega_x + \bar{\omega}_x$.

$$\dot{\omega}_x = a\omega_z\omega_y$$

$$\dot{\omega}_y = -b\left(\omega_x + \bar{\omega}_x\right)\omega_z$$

$$\dot{\omega}_z = c\left(\omega_x + \bar{\omega}_x\right)\omega_y$$



Rigid Body, Cont'd

Energy does not work for $\bar{\varpi}_x \neq 0$. Obvious?

So, how do we find Lyapunov function?

We want

$$V(0,0,0)=0,$$

$$V(\omega_x,\omega_y,\omega_z) > 0 \text{ if } (\omega_x,\omega_y,\omega_z) \neq (0,0,0) \text{ and } (\omega_x,\omega_y,\omega_z) \in D \text{ (some neighborhood ofthe origin)}$$

$$\dot{V} \leq 0$$

Lets look at all functions that satisfy $\dot{V} = 0$, i.e., that satisfy the pde:

$$\frac{\partial V}{\partial \omega_x} a\omega_z\omega_y + \frac{\partial V}{\partial \omega_y}\left(-b(\omega_x + \bar{\varpi}_x)\omega_z\right) + \frac{\partial V}{\partial \omega_z} c(\omega_x + \bar{\varpi}_x)\omega_y = 0$$

All solutions take the form:

$$f\left(\frac{b\omega_x^{\ 2} + 2b\omega_x\bar{\varpi}_x + a\omega_y^{\ 2}}{2a}, \frac{-c\omega_x^{\ 2} + 2c\omega_x\bar{\varpi}_x + a\omega_z^{\ 2}}{2a}\right)$$

$$V(\omega_x,\omega_y,\omega_z) = cA + bB + (cA - bB)^2 = \frac{1}{2}\left(\frac{8b^2c^2\bar{\varpi}_x^{\ 2}}{a^2}\omega_x^{\ 2} + c\omega_y^{\ 2} + b\omega_z^{\ 2}\right) + h.o.t.$$

(Arrows on the slide point from the first argument to A and from the second argument to B.)

Rigid Body, Cont'd

Clearly,

$$V(0)=0, V>0 \text{ on a neighborhood } D \text{ of } 0$$

$$\dot{V}=0$$

$\Rightarrow$ spin about x-axis is stability

- This is one approach to finding candidate Lyapunov functions
- The first order PDE usually has many solutions
- The method is connected to traditional 'first integral' methods to the study of stability in mechanics
- Same method can be used to prove stability for spin about z-axis, but spin about y-axis is unstable – why?

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LaSalle Invariance Theorem

Theorem : Suppose $V : R^n \to R$ is C^1 and let Ω_c denote a component of the region

$$\left\{ x \in R^n \middle| V(x) < c \right\}$$

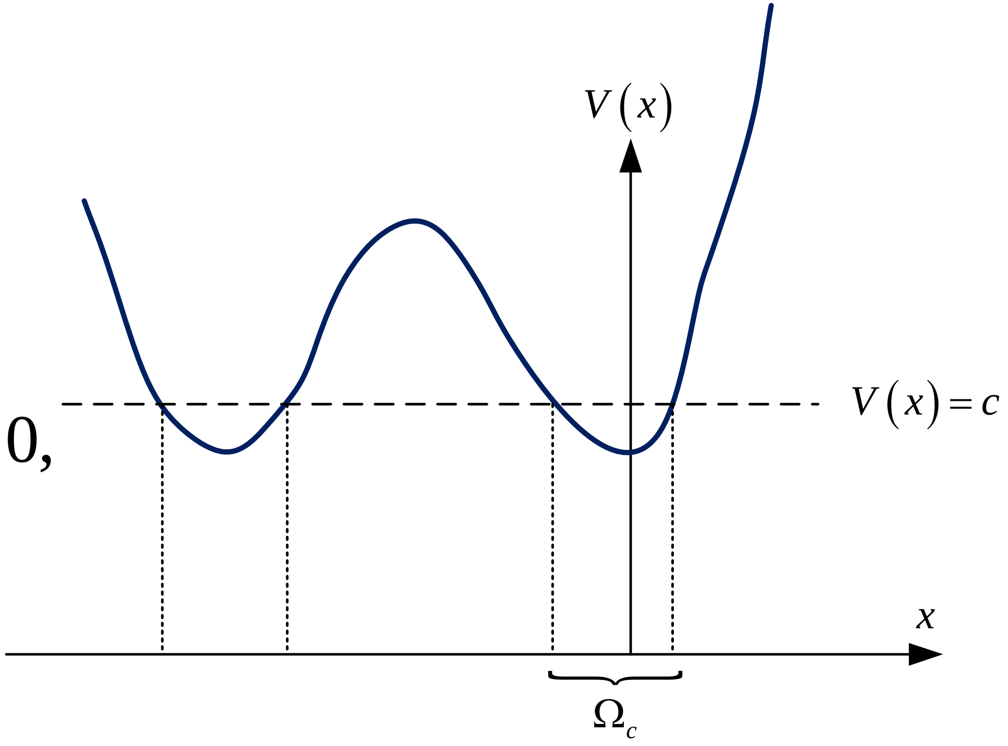
Suppose Ω_c is bounded and within $\Omega_c, \dot{V}(x) \le 0$.

Let E be the set of points within Ω_c where $\dot{V}(x) = 0$,

Let M be the largest invariant set within E.

$\Rightarrow$ every sol'n beginning in Ω_c tends to M.

as $t \to \infty$.



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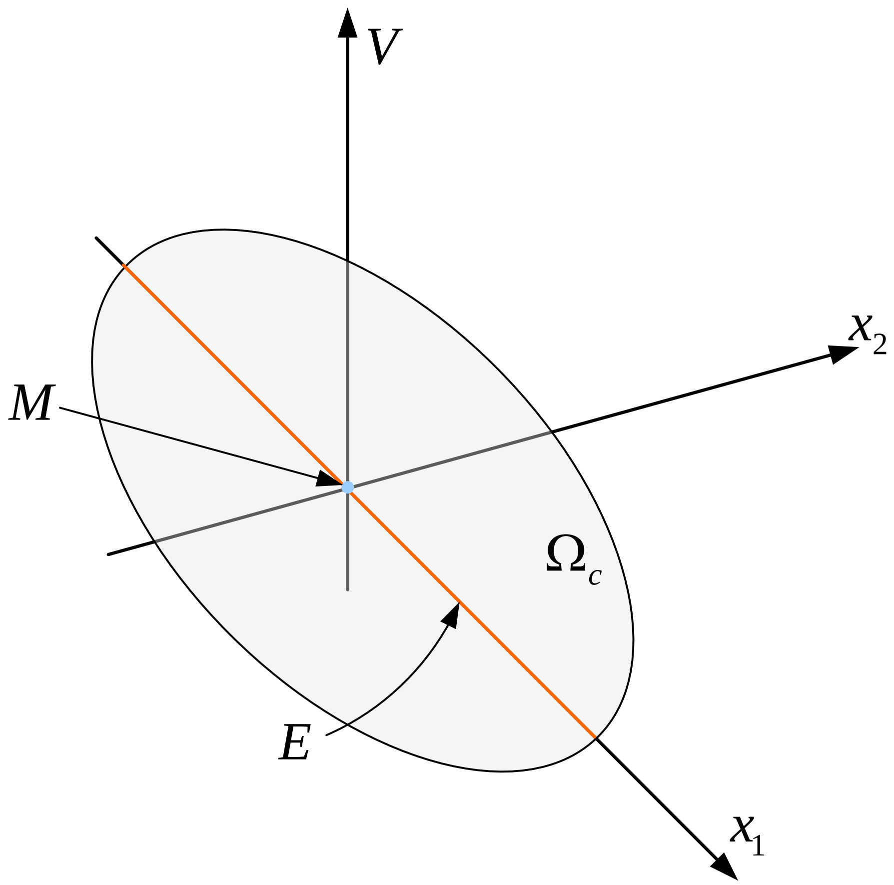
Example: LaSalle's Theorem

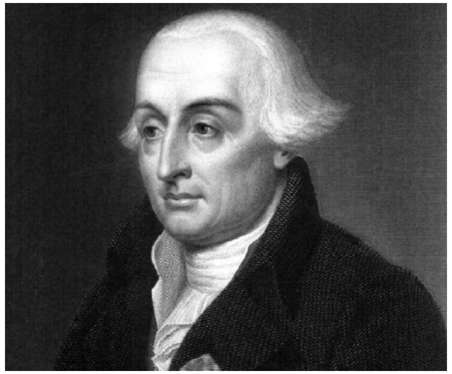
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -m^{-1}kx_1 - m^{-1}cx_2 \end{bmatrix},$$

$$V(x_1, x_2) = \frac{1}{2}mx_2^2 + \frac{1}{2}kx_1^2$$

$$\dot{V}(x_1, x_2) = -cx_2^2$$



Lagrangian Systems



$$\frac{d}{dt}\frac{\partial L(\dot{x},x)}{\partial \dot{x}} - \frac{\partial L(\dot{x},x)}{\partial x} = Q^T$$

$x \in R^n$ generalized coordinates

$\dot{x} = dx/dt$ generalized velocities

$L : R^{2n} \to R$ is the Lagrangian, $L(\dot{x},x) = T(\dot{x},x) - U(x)$

kinetic energy: $T(\dot{x},x) = \frac{1}{2}\dot{x}^T M(x)\dot{x}$

total energy: $V(x,\dot{x}) = T(\dot{x},x) + U(x)$

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Lagrange-Poincare Systems

$$\dot{x} = p, \quad \frac{d}{dt}\frac{\partial L(p,x)}{\partial p} - \frac{\partial L(p,x)}{\partial x} = Q^T$$

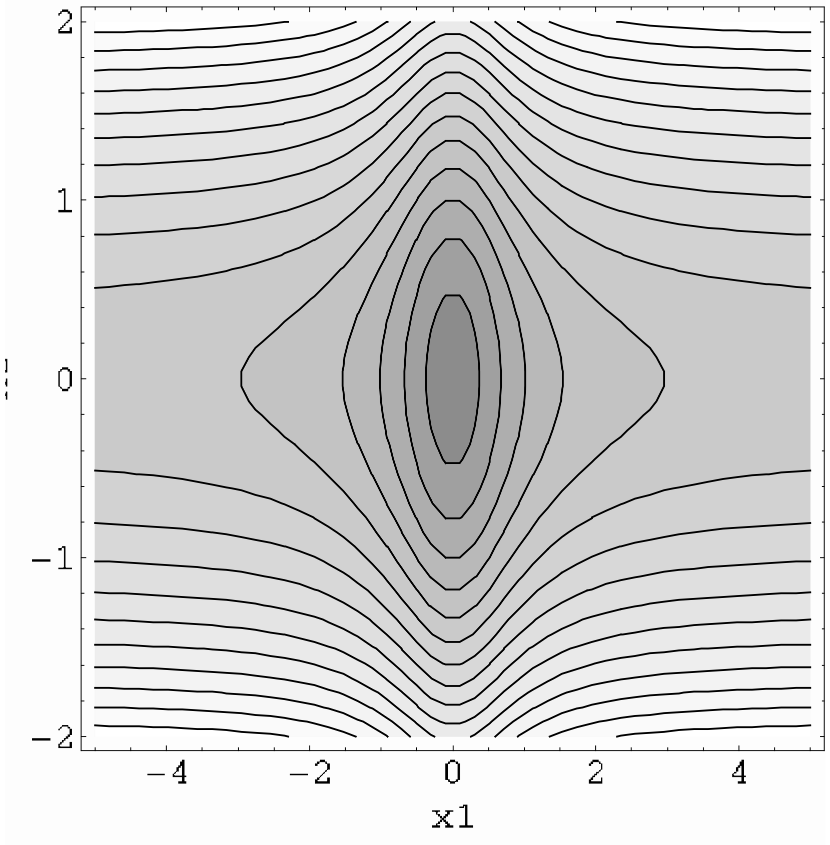
$$L(p,x) = T(p,x) - U(x), \quad T(p,x) = \tfrac{1}{2} p^T M(x) p$$

$$\frac{\partial L(p,x)}{\partial p} = p^T M(x), \quad \frac{d}{dt}\frac{\partial L(p,x)}{\partial p} = \dot{p}^T M(x) + p^T \frac{\partial p^T M(x)}{\partial x}$$

$$\dot{p}^T M(x) + p^T \frac{\partial p^T M(x)}{\partial x} - p^T \frac{\partial M(x) p}{\partial x} + \frac{\partial U(x)}{\partial x} = Q^T$$

$$\boxed{\begin{gathered} \dot{x} = p \\ M(x)\dot{p} + \left[\frac{\partial M(x) p}{\partial x} - \frac{\partial p^T M(x)}{\partial x}\right] p + \frac{\partial U(x)}{\partial x^T} = Q \end{gathered}}$$

Example



$$\dot{x}_1 = x_2, T = \frac{x_2^2}{2}, U = \frac{x_1^2}{1+x_1^2}$$

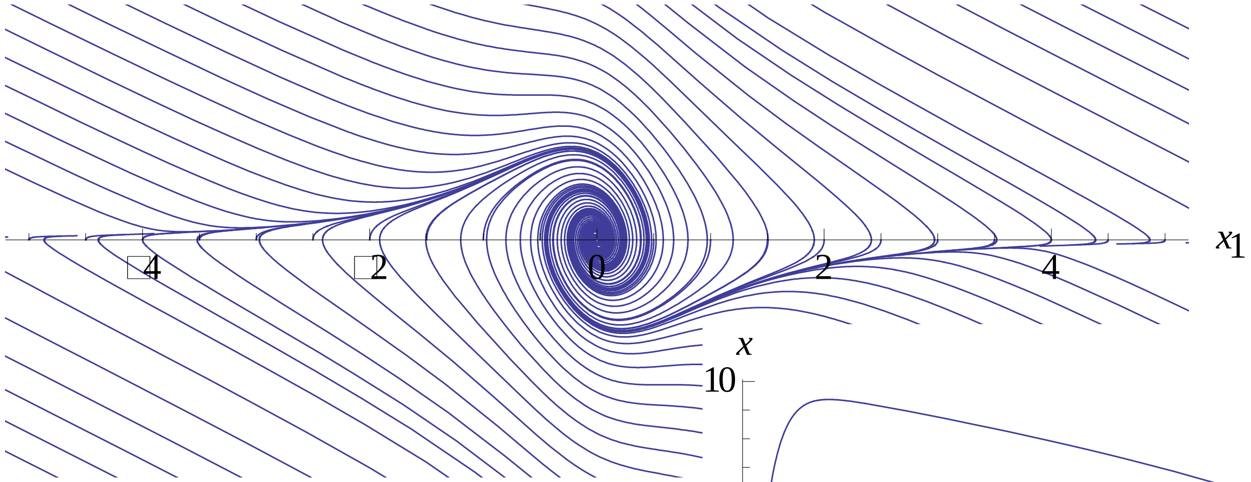
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\frac{x_1}{\left(1+x_1^2\right)^2} - cx_2 \end{bmatrix}$$

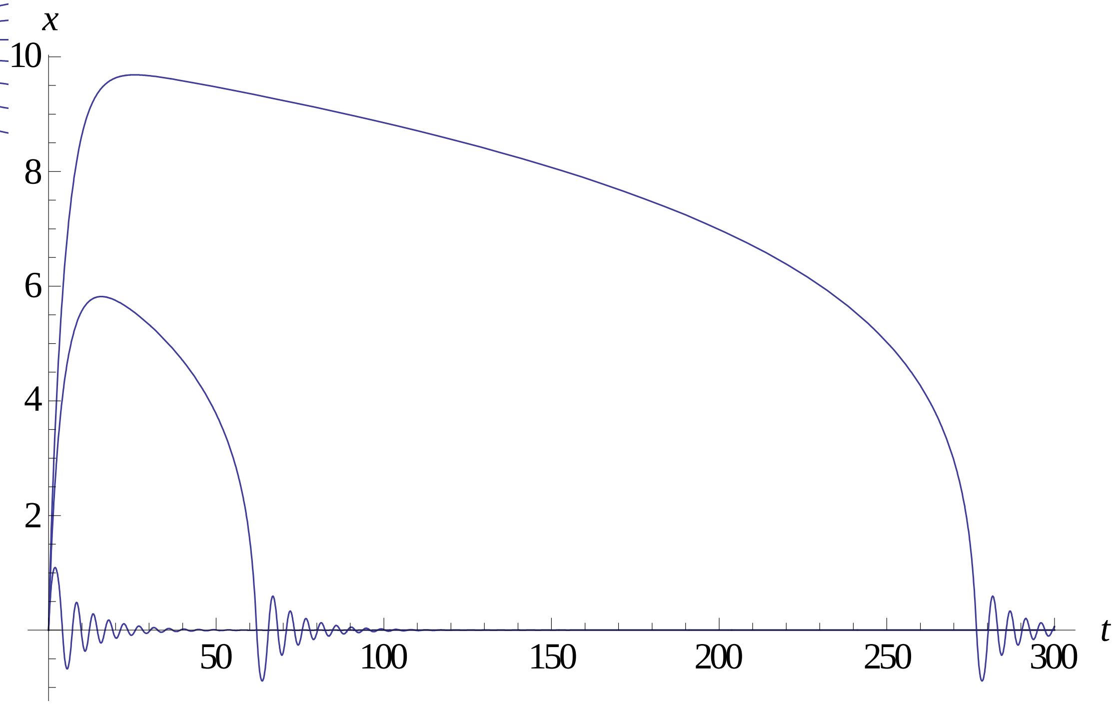
$$V(x) = \frac{x_2^2}{2} + \frac{x_1^2}{1+x_1^2}, \dot{V}(x) = -cx_2 \leq 0 \text{ for } c > 0$$

Notice that

- the level sets are unbounded for $V(x) = \text{constant} \geq 1$
- $V(x)$ is not radially unbounded

Example, Cont'd





First Integrals

Definition : A *first integral* of the differential equation

$$\dot{x} = f(x,t)$$

is a scalar function $\varphi(x,t)$ that is constant along trajectories, i.e.,

$$\dot{\varphi}(x,t) = \frac{\partial \varphi(x,t)}{\partial x} f(x,t) + \frac{\partial \varphi(x,t)}{\partial t} \equiv 0$$

Observation : For simplicity, consider the autonomous case $\dot{x} = f(x)$. Suppose $\varphi_1(x)$ is a first integral and $\varphi_2(x),\ldots,\varphi_n(x)$ are arbitrary independent functions on a neighborhhod of the point x_0, i.e.,

$$\det \frac{\partial}{\partial x}\begin{bmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{bmatrix}_{x=x_0} \neq 0$$

Then we can define coordinate transformation $x \to z$, via

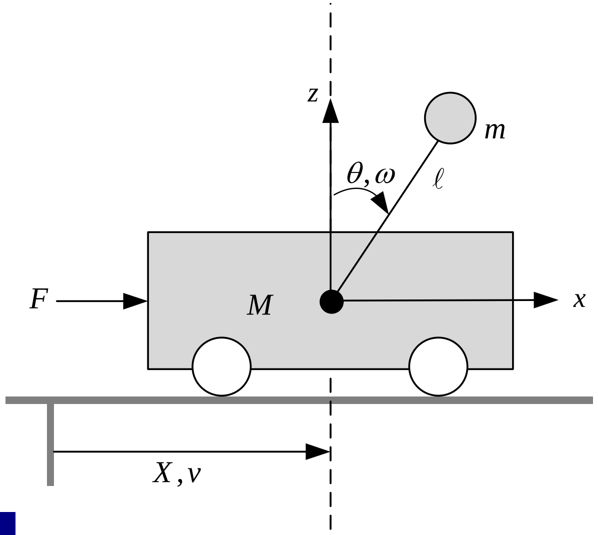
$$z = \varphi(x) \Rightarrow \dot{z} = \left[\frac{\partial \varphi(x)}{\partial x} f(x)\right]_{x=\varphi^{-1}(z)} \Rightarrow \dot{z}_1 = 0 \Rightarrow z_1 \equiv \text{constant}$$

The problem has been reduced to solving $n-1$ differential equations.

Noether's Theorem

If the Lagrangian is invariant under a smooth 1 parameter change of coordinates, $h_s : M \to M,\ s \in R$, then the Lagrangian system has a first integral

$$\Phi(p,q) = \frac{\partial L}{\partial p} \frac{dh_s(q)}{ds}\bigg|_{s=0}$$



$$T(p,q) = \frac{1}{2}\begin{bmatrix} v & \omega \end{bmatrix}\begin{bmatrix} M+m & m\ell\cos\theta \\ m\ell\cos\theta & m\ell^2 \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix}, \quad U(q) = -mg\ell\cos\theta$$

$$\begin{bmatrix} M+m & m\ell\cos\theta \\ m\ell\cos\theta & m\ell^2 \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 & -m\ell\omega\sin\theta \\ \tfrac{1}{2}m\ell\omega\sin\theta & \tfrac{1}{2}m\ell v\sin\theta \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ -mg\ell\sin\theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$h_s(X,\theta) = \begin{bmatrix} X+s \\ 0 \end{bmatrix}, \quad \Phi(p,q) = (M+m)v$$

Momentum in X direction

Chetaev's Method

Consider the system of equations

$$\dot{x} = f(x,t), \quad f(0,t) = 0$$

We wish to study the stability of the equilibrium point $x = 0$.

Obviously, if $\varphi(x,t)$ is a first integral and it is also a positive definite function, then $V(x,t) = \varphi(x,t)$ establishes stability. But suppose $\varphi(x,t)$ is not positive definite?

Suppose the system has k first integrals $\varphi_1(x,t),\ldots,\varphi_k(x,t)$ such that $\varphi_i(0,t) = 0$.

Chetaev suggested the construction of Lyapunov functions of the form:

$$V(x,t) = \sum_{i=1}^{k} \alpha_i \varphi_i(x,t) + \sum_{i=1}^{k} \beta_i \varphi_i^2(x,t)$$

Chetaev Instability Theorem

Let D be a neighborhood of the origin. Suppose there is a function $V(x): D \to R$ and a set $D_1 \subset D$ such that

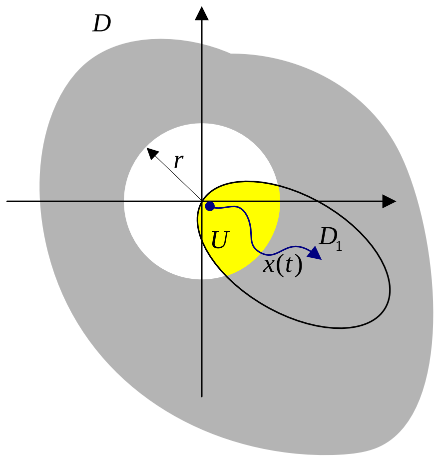
1) $V(x)$ is C^1 on D,

2) the origin belongs to the boundary of D_1, ∂D_1,

3) $V(x) > 0, \dot{V}(x) > 0$ on D_1,

4) on the boundary of D_1 inside D, i.e., on $\partial D_1 \cap D, V(x) = 0$

Then the origin is unstable.



Example, Rigid Body, Cont'd

Consider the rigid body with spin about the y-axis (intermediate inertia), $\bar{\varpi} = \left(0, \bar{\varpi}_y, 0\right)^T$

Shifted equations:
$$\begin{aligned} \dot{\omega}_x &= a\omega_z\left(\omega_y + \bar{\varpi}_y\right) \\ \dot{\omega}_y &= -b\omega_x\omega_z \\ \dot{\omega}_z &= c\left(\omega_y + \bar{\varpi}_y\right)\omega_x \end{aligned}$$

Attempts to prove stability fails. So, try to prove instability.

Consider $V\left(\omega_x, \omega_y, \omega_z\right) = \omega_x\omega_z$

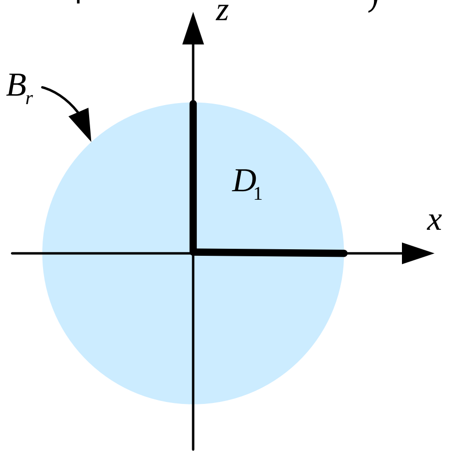
Let $B_r = \left\{\left(\omega_x, \omega_y, \omega_z\right) \middle| \omega_x^2 + \omega_y^2 + \omega_z^2 < r^2\right\}$ and $D_1 = \left\{\left(\omega_x, \omega_y, \omega_z\right) \in B_r \middle| \omega_x > 0, \omega_z > 0\right\}$

so that $V > 0$ on D_1 and $V = 0$ on ∂D_1

$\dot{V} = a\omega_z^2\left(\omega_y + \bar{\varpi}_y\right) + c\left(\omega_y + \bar{\varpi}_y\right)\omega_x^2 = \left(\omega_y + \bar{\varpi}_y\right)\left(a\omega_z^2 + c\omega_x^2\right)$

We can take $r^2 < \bar{\varpi}_y^2 \Rightarrow \omega_y + \bar{\varpi}_y > 0 \forall \left(\omega_x, \omega_y, \omega_z\right) \in B_r$

in which case $\dot{V} > 0$ on $D_1 \Rightarrow$ instability



Stability of Linear Systems - Summary

Consider the linear system

$$\dot{x} = Ax$$

Choose $V(x) = x^T P x$

$$\Rightarrow \dot{V}(x) = x^T \left(A^T P + PA\right) x := -x^T Q x$$

sufficient condition

a) if their exists a positive definite pair of matrices P, Q

that satisfy

$$A^T P + PA = -Q \quad \text{(Lyapunov equation)}$$

the origin is asymptotically stable.

b) if P has at least one negative eigenvalue and $Q > 0$, the

origin is unstable.

necessary condition

c) if the origin is asymptotically stable then for any $Q > 0$,

there is a unique solution, $P > 0$, of the Lyapunov equation.

Second Order Systems

Consider the system

$$M\ddot{x} + C\dot{x} + Kx = 0,\ M^T = M > 0,\ C^T = C > 0,\ K^T = K > 0$$

$$\boxed{E(\dot{x}, x) = \tfrac{1}{2}\dot{x}^T M\dot{x} + \tfrac{1}{2}x^T Kx}$$

$$\frac{d}{dt}E(\dot{x}, x) = \dot{x}^T M\ddot{x} + x^T K\dot{x} = -\dot{x}^T\left[C\dot{x} + Kx\right] + x^T K\dot{x}$$

$$\boxed{\frac{d}{dt}E(\dot{x}, x) = -\dot{x}^T C\dot{x}}$$

Some interesting generalizations:

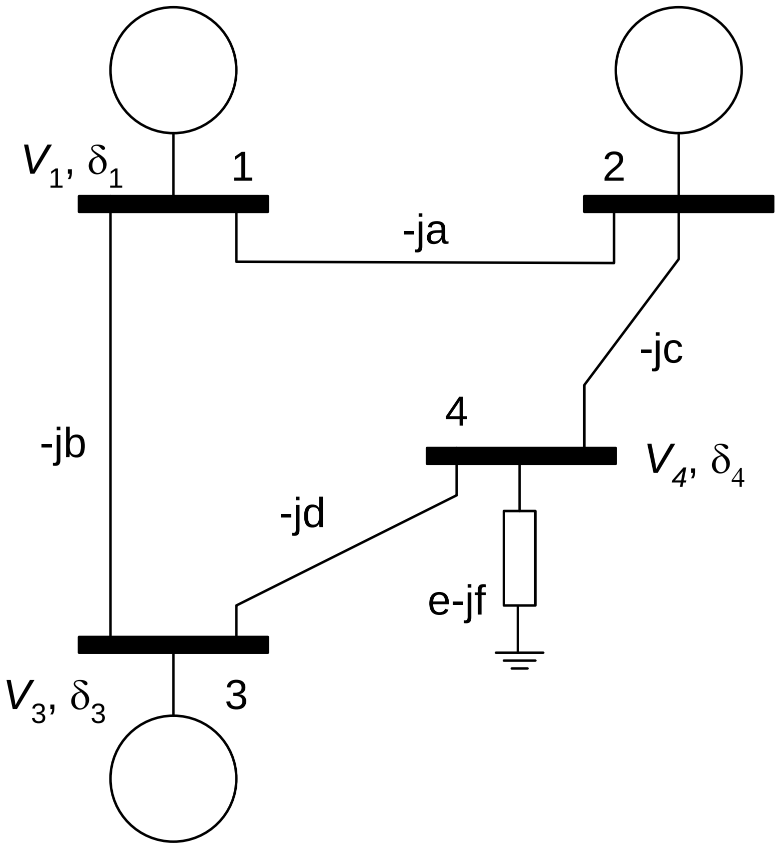
$$1)\ C \geq 0,\quad 2)\ C^T \neq C,\quad 3)\ K^T \neq K$$

The anti-symmetric terms correspond to 'circulatory' forces (transfer conductances in power systems) – they are non-conservative.

The anti-symmetric terms correspond to 'gyroscope' forces – they are conservative.

Example

- Assume uniform damping
- Assume e=0
- Designate Gen 1 as swing bus
- Eliminate internal bus 4



$$\ddot{\theta}_1 + \gamma\dot{\theta}_1 = \Delta P_1 - b_{13}\sin(\theta_1) - b_{12}\sin(\theta_1 - \theta_2)$$

$$\ddot{\theta}_2 + \gamma\dot{\theta}_2 = \Delta P_2 + b_{12}\sin(\theta_1 - \theta_2) - b_{23}\sin(\theta_2)$$

$$\theta_1 = \delta_2 - \delta_1, \theta_2 = \delta_3 - \delta_1, \Delta P_1 = P_2 - P_1, \Delta P_2 = P_3 - P_1$$

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Example Cont'd

- This is a Lagrangian system with

$$U(\theta_1,\theta_2) = -\Delta P_1\theta_1 - \Delta P_2\theta_2 - b_{13}\cos(\theta_1) - b_{12}\cos(\theta_1-\theta_2) - b_{23}\cos(\theta_2)$$

$$T(\omega_1,\omega_2) = \frac{1}{2}(\omega_1^2+\omega_2^2), \quad Q = [-\gamma\omega_1 \quad -\gamma\omega_2]$$

- To study stability choose total energy as Lyapunov function

$$V = T(\omega_1,\omega_2) + U(\theta_1,\theta_2)$$

$$\dot{V} = -\gamma\omega_1^2 - \gamma\omega_2^2 \le 0$$

Note: $T(0,0)=0$ and $T(\omega_1,\omega_2)>0\ \forall(\omega_1,\omega_2)\neq 0 \quad \Rightarrow$

Equilibria corresponding to $U(\theta_1,\theta_2)$ a local minimum are stable.

G. V. Aronovich and N. A. Kartvelishvili, "Application of Stability Theory to Static and Dynamic Stability Problems of Power Systems," presented at Second All-union Conference on Theoretical and Applied mechanics, Moscow, 1965.

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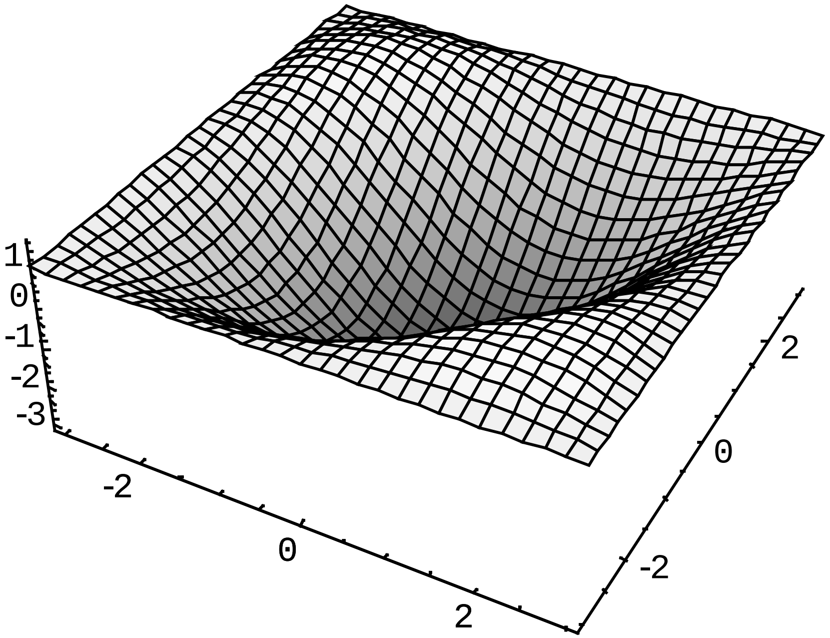
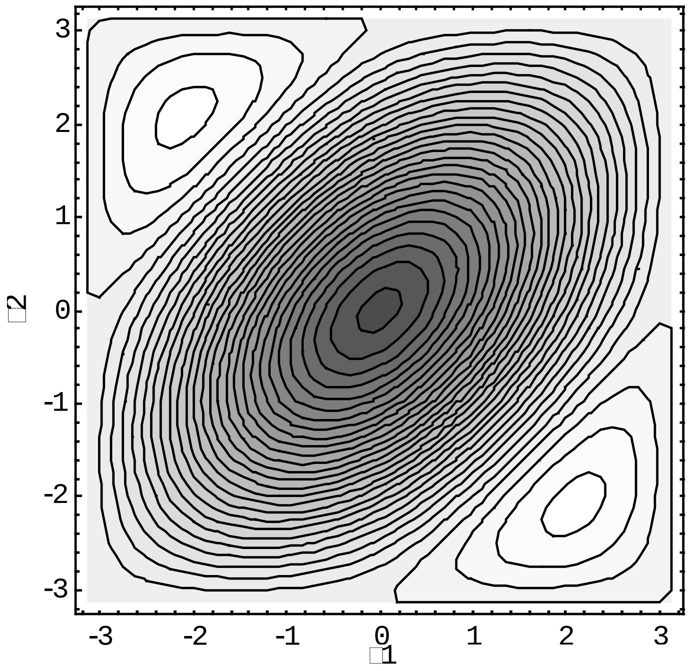
Example, Cont'd

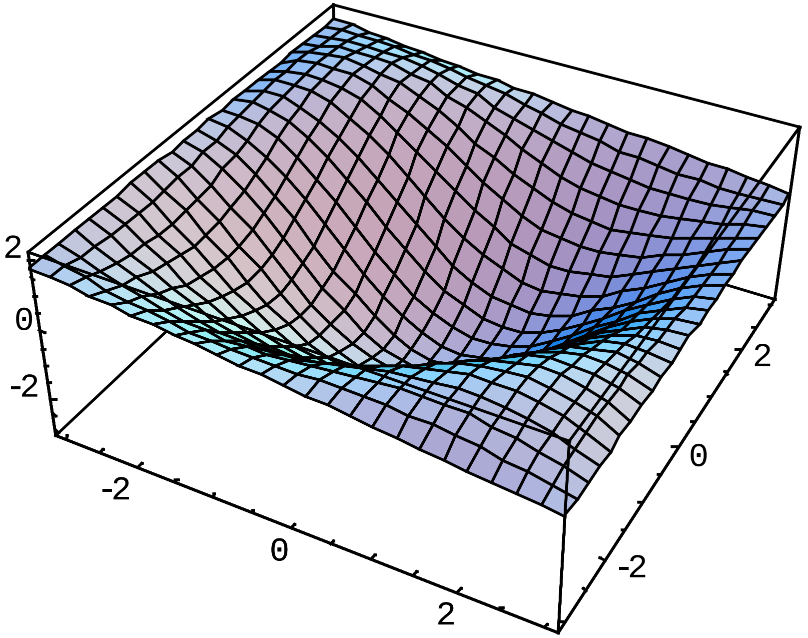
Since $-\pi \le \theta_1 < \pi$ and $-\pi \le \theta_2 < \pi$ we should consider $U\left(\theta_1, \theta_2\right)$ as a function on a torus $U : \mathscr{T} \to R$

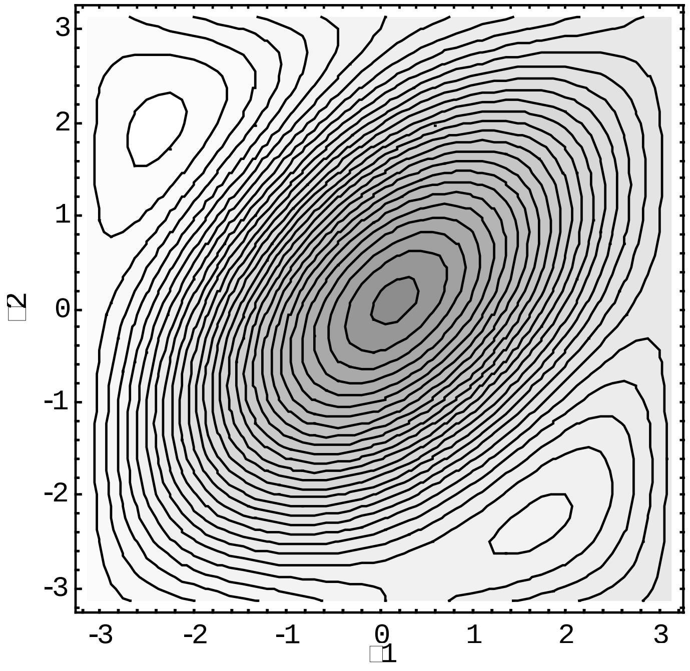




$$P_1 = 0,\ P_2 = 0,\ b_{12} = 1,\ b_{13} = 1,\ b_{23} = 1$$

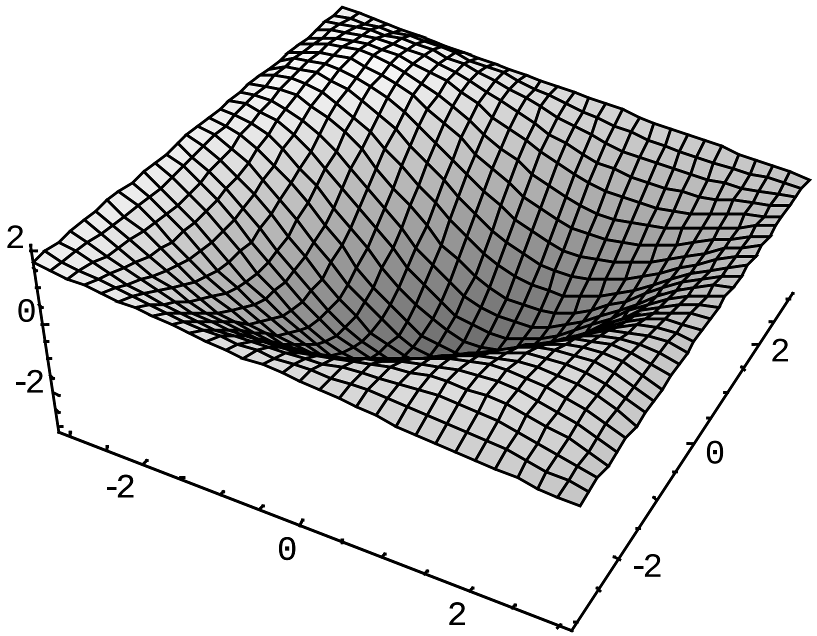
Example, Cont'd

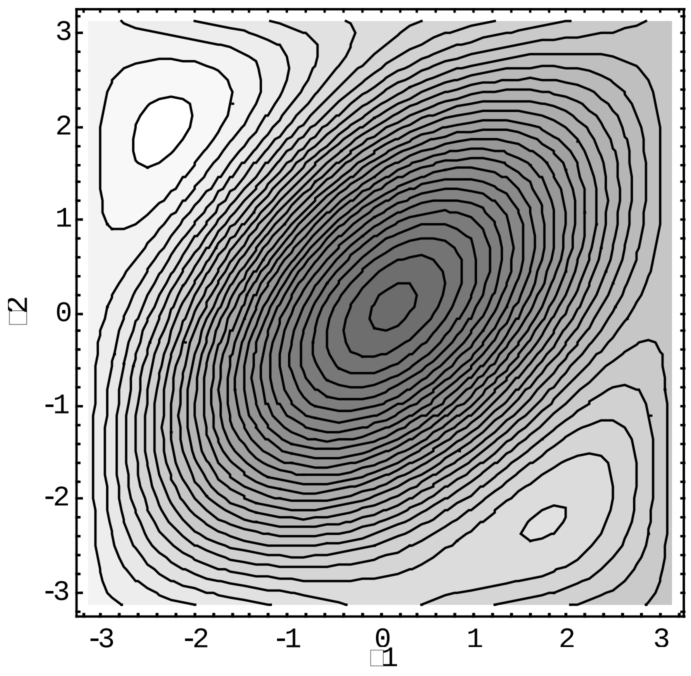




$$P_1 = .25,\ P_2 = 0,\ b_{12} = 1,\ b_{13} = 1,\ b_{23} = 1$$

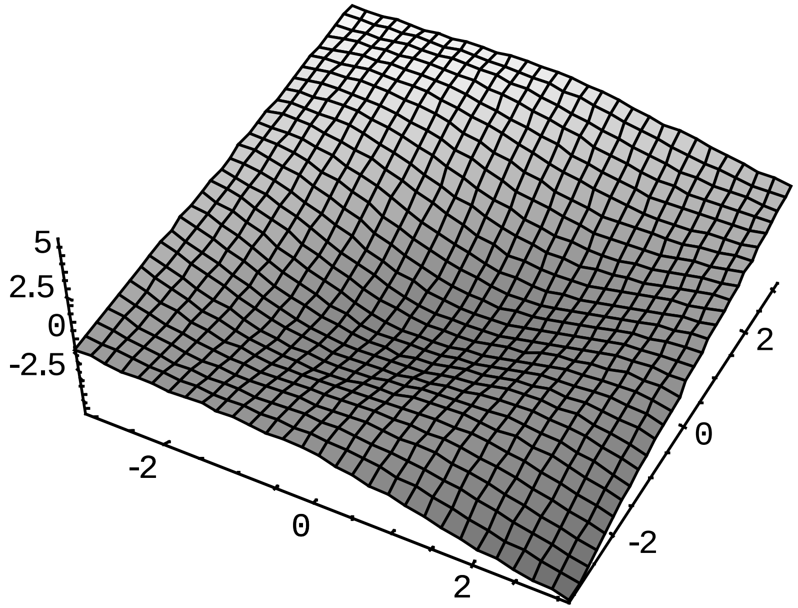
Example Cont'd

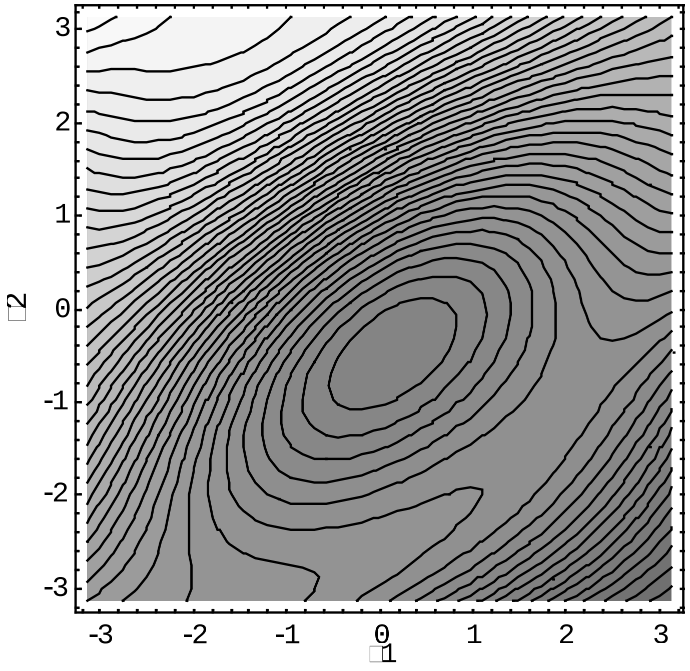




$$P_1 = \pi/15,\ P_2 = 0,\ b_{12} = 1,\ b_{13} = 1,\ b_{23} = 1$$

Example Cont'd





$$P_1 = \pi / 5,\ P_2 = -1,\ b_{12} = 1,\ b_{13} = .5,\ b_{23} = 1$$

Control Lyapunov Function

Consider the controlled system

$$\dot{x} = f(x,u), \quad x \in D \subset R^n \text{ containing x=0}, u \in R^m$$

Find $u(x)$ such that all trajectories beginning in D converge to $x = 0$.

Definition: A control Lyapunov function (CLF) is a function $V : D \to R$ with $V(0) = 0, \; V(x) > 0, \; x \neq 0,$ such that

$$\forall x \neq 0 \in D, \exists u(x) \quad \dot{V}(x) = \frac{\partial V}{\partial x} f(x, u(x)) < 0$$

Theorem: (Artsteins Theorem) A differentiable CLF exists iff there exists a 'regular' feedback control $u(x)$.

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Example

Consider the nonlinear mass-spring-damper system:

$$v = \dot{q}, \qquad m\left(1+q^2\right)\dot{v} + bv + k_0 q + k_1 q^3 = u$$

$$\frac{d}{dt}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} v \\ \dfrac{1}{m\left(1+q^2\right)}\left(-k_0 q - k_1 q^3 - bv + u\right) \end{bmatrix}$$

suppose the target state is $v = 0,\ q = 0,$

Define $r^2 = v^2 + \alpha q^2,\ \alpha > 0$. A CLF candidate is $V = \dfrac{1}{2} r^2$. This is positive definite wrt the target state

$$\dot{V} = r\dot{r} = \left(\dot{v} + \alpha q\right)v = v\left\{\frac{u - bv - k_0 q - k_1 q^3}{m\left(1+q^2\right)} + \alpha q\right\} = -\kappa v^2, \quad \kappa > 0$$

$$\left\{\frac{u - bv - k_0 q - k_1 q^3}{m\left(1+q^2\right)} + \alpha q\right\} = -\kappa v \le 0 \quad \Rightarrow \boxed{u = bv + k_0 q + k_1 q^3 - m\left(1+q^2\right)\left(\alpha q + \kappa v\right)}$$

$$\Rightarrow \text{ closed loop } \quad \frac{d}{dt}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\kappa v - \alpha q \end{bmatrix}$$