Vector Fields, Flows, & Distributions

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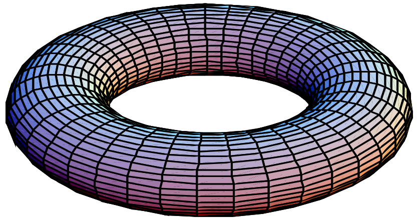

Outline

- Vector fields & flows
- Lie Bracket
- Distributions
- Controllability
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Vector Fields

Definition: A *vector field* v on M is a map which assigns to each point $p \in M$, a tangent vector $v(p) \in TM_p$. It is a C^k-*vector field* if for each $p \in M$ there exist local coordinates (U, φ) such that each component $v_i(x)$, i=1,..,m is a C^k function for each $x \in \varphi(U)$.

Definition: An *integral curve* of a vector field v on M is a parameterized curve $p = \phi(t)$, $t \in (t_1, t_2) \subset R$ whose tangent vector at any point coincides with v at that point.

In local coordinates, the vector field is written as

$$\text{a vector } v(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \\ \vdots \\ v_n(x) \end{bmatrix}$$

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Flow Function

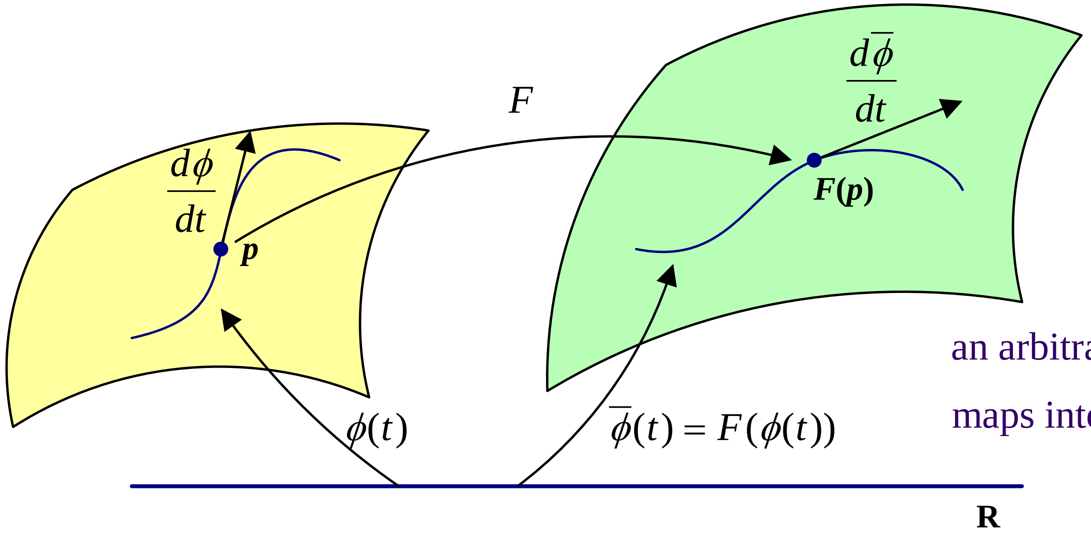
Definition: Let *v* be a smooth vector field on M and denote the parameterized maximal integral curve through p∈M by Ψ(t,p) and Ψ(0,p)=p. Ψ(t,p) is called the *flow generated by v.*

<u>Properties of flows:</u>

• satisfies ode $$\frac{d}{dt}\Psi(t,p)=v\big(\Psi(t,p)\big), \quad \Psi(0,p)=p$$

• semigroup property $$\Psi\big(t_2,\Psi(t_1,p)\big)=\Psi(t_1+t_2,p)$$

Differential Map



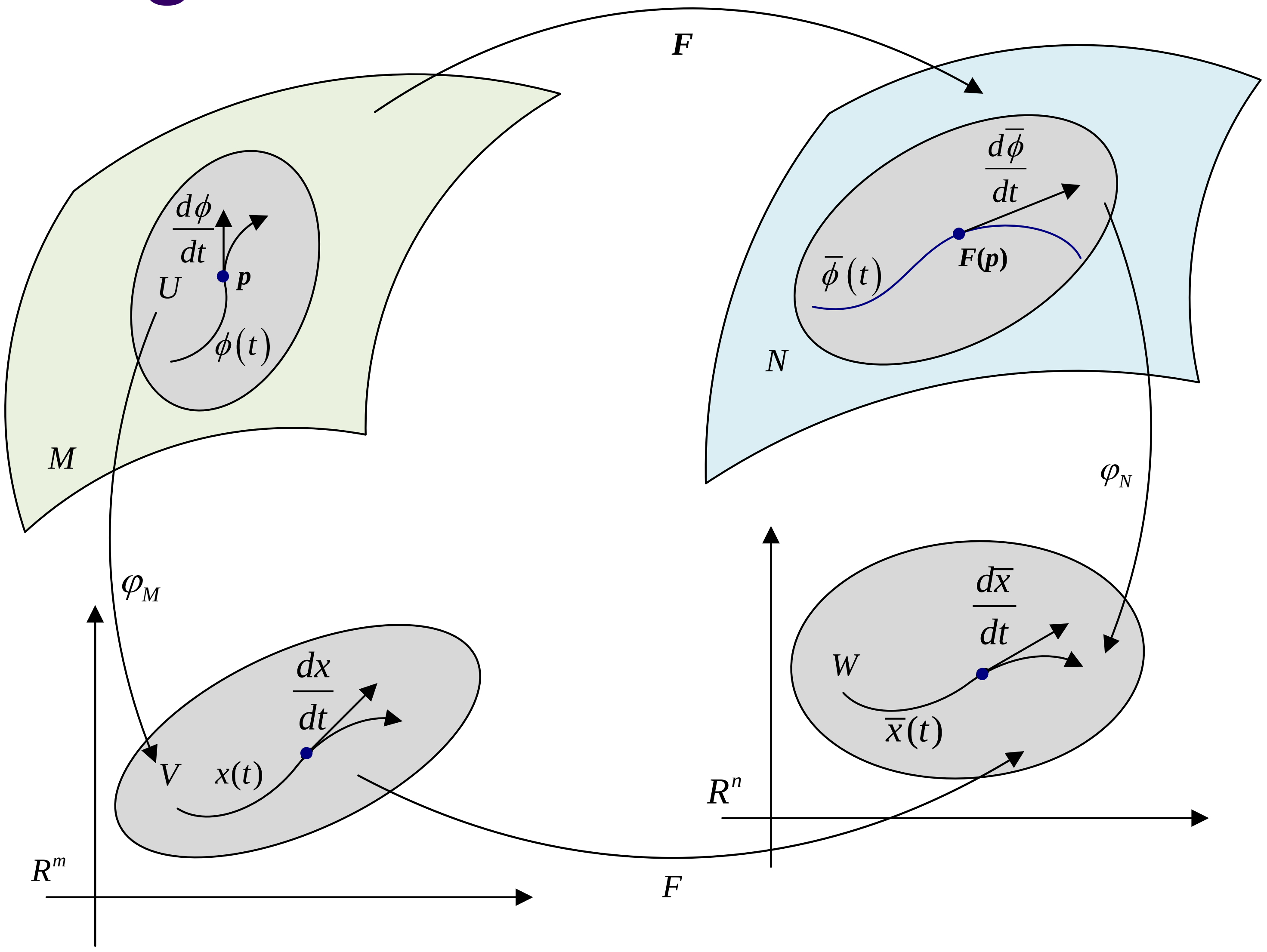
an arbitray curve $\phi(t)$ on M passing through point p
maps into $\bar{\phi}(t)$ on N passing through point $F(p)$

Given the map $F: M \rightarrow N$, the *differential map* is the induced mapping

$$F_*: TM_p \rightarrow TN_{F(p)}$$

that takes tangent vectors into tangent vectors.

Analysis using Local Coordinates



Lie Derivative

Definition: Let v(x) denote a vector field on M and F(x) a mapping from M to R^n, both in local coordinates. Then the *Lie derivative of order* 0,…,*k is*

$$L_v^0(F) = F, \quad L_v^k(F) = \frac{\partial L_v^{k-1}}{\partial x} v$$

Example:

$$F(x) = Bx,\ B \in R^{m\times n}, v(x) = Ax,\ A \in R^{m\times m}$$

$$L_v^0(F) = Bx,\ L_v^1(F) = \frac{\partial Bx}{\partial x} Ax = BAx$$

$$L_v^2(F) = \frac{\partial BAx}{\partial x} Ax = BA^2 x, \quad L_v^3(F) = \frac{\partial BA^2 x}{\partial x} Ax = BA^3 x$$

Series Expansion Along Trajectory

Suppose $x(t)$ satisfies $\dot{x} = v(x)$, $x(0) = x_0$. Let $f : R^n \to R^p$ be any smooth function

$$f(x(t)) = f(x_0) + \left[\frac{d}{dt} f(x(t))\right]_{t=0} t + \frac{1}{2}\left[\frac{d^2}{dt^2} f(x(t))\right]_{t=0} t^2 + \cdots$$

$$\frac{d}{dt} f(x(t)) = \frac{\partial f}{\partial x} v(x) = L_v^1(f)$$

$$\frac{d^2}{dt^2} f(x(t)) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} v(x)\right) v(x) = L_v^2(f)$$

$$f(x(t)) = f(x_0) + L_v^1(f)\Big|_{x=x_0} t + \frac{1}{2} L_v^2(f)\Big|_{x=x_0} t^2 + \cdots$$

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Series Representation of Exp Map

Thus, for any smooth f

$$f\left(x(t)\right)=\sum_{k=0}^{\infty}\frac{t^k}{k!}L_v^k(f)\Big|_{x=x_0}$$

For $f(x)=x$

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}L_v^k(x)\Big|_{x=x_0}\quad\Rightarrow\Psi(t,x_0)=\sum_{k=0}^{\infty}\frac{t^k}{k!}L_v^k(x)\Big|_{x=x_0}$$

which motivates the notation

$$\Psi(t,x_0)=e^{tv}x_0=\sum_{k=0}^{\infty}\frac{t^k}{k!}L_v^k(x)\Big|_{x=x_0}$$ or, more simply $e^{tv}x_0=\sum_{k=0}^{\infty}\frac{t^k}{k!}L_v^k(x_0)$

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Exponential Map Properties

We have adopted the notation

$$e^{t\mathbf{v}} x_0 := \Psi\left(t, x_0\right)$$

The motivation for this is that the flow satisfies the three basic properties ordinarily associated with exponentiation – from properties of $\Psi(t,p)$.

$e^{0 \cdot v} x_0 = x_0$	boundary condition
$\frac{d}{dt} e^{t v} x_0 = v\left(e^{t\mathbf{v}} x_0\right)$	differential equation
$e^{(t_1 + t_2) v} x_0 = e^{t_2 v} e^{t_1 v} x_0$	semi-group property

Example: general linear field

$$v(x) = Ax$$

$$L_v^0(x) = x, L_v^1(x) = Ax, L_v^2(x) = A^2x, L_v^3(x) = A^3x, \dots$$

$$\Downarrow$$

$$e^{tv}x_0 == \sum_{k=0}^{\infty} \frac{t^k}{k!} L_v^k(x)\Big|_{x=x_0} = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x_0 \right) = e^{At}x_0$$

Example: Affine Field

$$v(x) = Ax + b$$

$$L_v^1(x) = Ax + b$$

$$L_v^k(x) = L_v^1\left(A^{k-1}x + A^{k-2}b\right) = A^k x + A^{k-1}b$$

$$e^{t\mathbf{v}}x = \sum_{k=0}^{\infty}\frac{t^k}{k!}L_v^k(x) = \left(\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k x + \sum_{k=0}^{\infty}\frac{t^k}{k!}A^{k-1}b\right) = e^{At}x + e^{At}A^{-1}b$$

Lie Bracket

Definition: If *v*,*w* are vector fields on M, then their *Lie bracket* [*v*,*w*] is the unique vector field defined in local coordinates by the formula

$$[v, w] = \frac{\partial w}{\partial x} v - \frac{\partial v}{\partial x} w$$

Property:

$$\left. \frac{dw\left(\Psi\left(t, x\right)\right)}{dt} \right|_{t=0} = \left.\left[v, w\right]\right|_x$$

The rate of change of *w* along the flow of *v*

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Lie Bracket Interpretation

Let us consider the Lie bracket as a commutator of flows. Beginning at point x in M follow the flow generated by v for an infinitesimal time which we take as $\sqrt{\varepsilon}$ for convenience. This takes us to point

$$y = \exp(\sqrt{\varepsilon}\ \mathbf{v})x$$

Then follow **w** for the same length of time, then **-v**, then **-w**. This brings us to a point ψ given by

$$\psi(\varepsilon, x) = e^{-\sqrt{\varepsilon}\mathbf{w}}\, e^{-\sqrt{\varepsilon}\mathbf{v}}\, e^{\sqrt{\varepsilon}\mathbf{w}}\, e^{\sqrt{\varepsilon}\mathbf{v}}\, x$$

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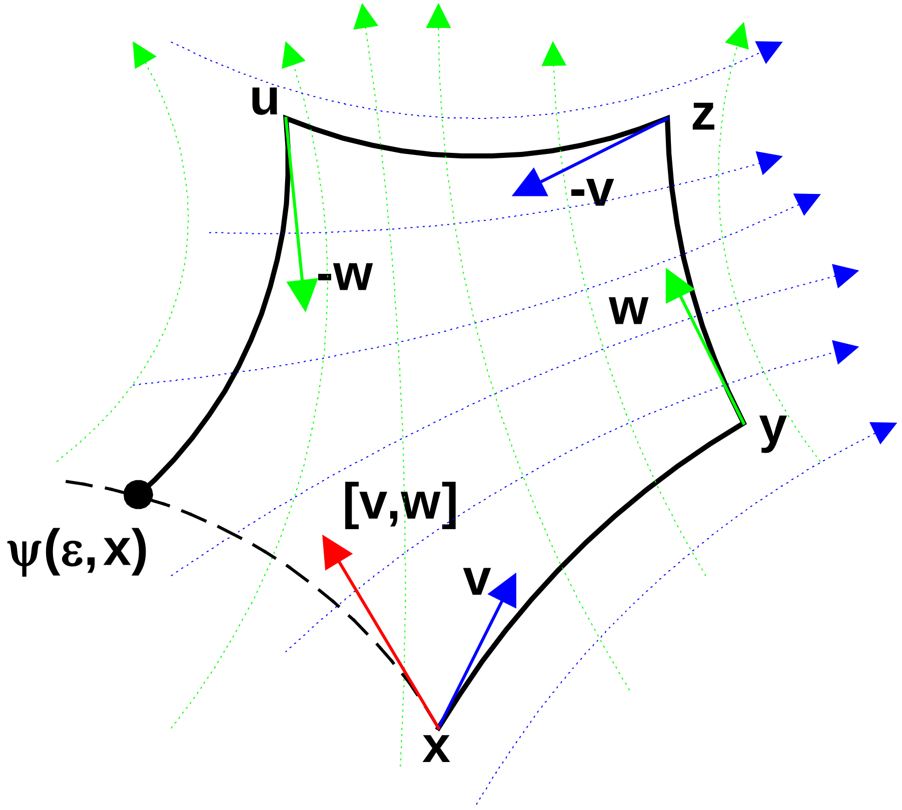
Lie Bracket Interpretation Continued



$$\frac{d}{d\varepsilon}\Psi\left(0^{+},x\right)=\left[v,w\right]\Big|_{x}$$

Iterated Lie Bracket

We recursively define higher order Lie Brackets:

$$ad_v^0(w) = w$$

$$ad_v^k = \left[v, ad_v^{k-1}(w)\right]$$

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Distributions

$$v_1, \ldots, v_r \text{ is a set of vector fields on } M$$

$$\Delta_p = \text{span}\{v(p)_1, \ldots, v_r(p)\} \text{ is a subspace of } TM_p$$

Definition: A *smooth distribution* Δ on M is a map which assigns to each point $p \in M$, a subspace of the tangent space to M at p, $\Delta_p \subset TMp$ such that Δ_p is the span of a set of smooth vector fields $v_1,..,v_r$ evaluated at p. We write $\Delta = \text{span}\{v_1,..,v_r\}$.

Definition: An *integral submanifold* of a set of vector fields $v_1,..,v_r$ is a submanifold $N \subset M$ whose tangent space TNp is spanned by $\{v_1(p),..,v_r(p)\}$ for each $p \in N$. The set of vector fields is *(completely) integrable* if through every point $p \in M$ there passes an integral submanifold.

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Involutive Distributions

Definition: A system of smooth vector fields $\{v_1,..,v_r\}$ on M is *in involution* if there exist smooth real valued functions $c_k^{ij}(p)$, $p \in M$ and $i,j,k = 1,..,r$ such that for each i,j

$$\left[v_i , v_j \right] = \sum_{k=1}^{r} c_k^{ij} v_k$$

Proposition: (Froebenius) Let $\{v_1,..,v_r\}$ be an involutive system of vector fields with dim $[\text{span}\{v_1,..,v_r\}]=k$ on M. Then the system is integrable with all integral manifolds of dimension k.

Proposition: (Hermann) Let $\{v_1,..,v_r\}$ be a system of smooth vector fields on M. Then the system is integrable if and only if it is in involution.

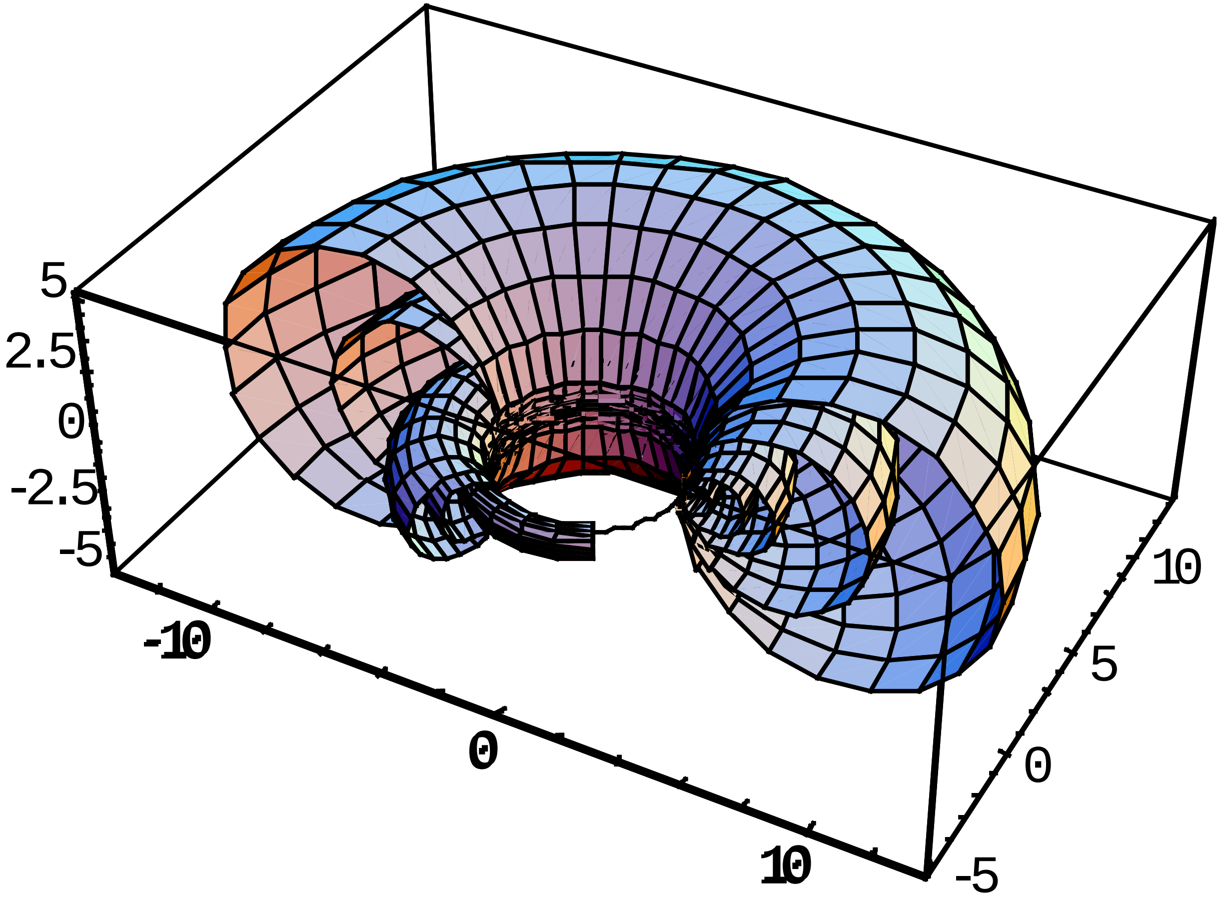
Example

$$M = R^3 \qquad \Delta = \text{span}\{v, w\} \qquad v = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 2zx \\ 2yz \\ z^2 + 1 - x^2 - y^2 \end{bmatrix}$$

$[v, w] \equiv 0$ so the distribution Δ is completely integrable. The distribution is singular because $\dim \Delta = 2$ everywhere except on the z-axis $x = 0, y = 0$ and on the circle $x^2 + y^2 = 1, z = 0$ where $\dim \Delta = 1$ The *z*-axis and the circle are one-dimensional integral manifolds. All others are the tori:

$$T_c = \left\{ (x, y, z) \in R^3 \middle| (x^2 + y^2)^{-1/2} (x^2 + y^2 + z^2 + 1) = c > 2 \right\}$$

Example



Invariant Distributions

Definition: A distribution $\Delta = \{v_1, \ldots, v_r\}$ on M is *invariant* with respect to a vector field f on M if the Lie bracket $[f,v_i]$, for each $i = 1, \ldots, r$ is a vector field of Δ.

Notation: $[f, \Delta] = \text{span}\{[f, v_i], i = 1, \ldots, r\}$

that Δ is invariant with respect to f may be stated $[f,\Delta] \subset \Delta$.

In general

$$\Delta + [f,\Delta] = \Delta + \text{span}\{[f,v_i], i=1,..,r\} = \text{span}\{v_1,..,v_r,[f,v_1],..,[f,v_r]\}$$

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Involutive Closure ~ 1

- Problem 1: find the smallest distribution with the following properties
  - It is nonsingular
  - It contains a given distribution Δ
  - It is involutive
  - It is invariant w.r.t. a given set of vector fields, $\tau_1,\ldots,\tau_q$

$$\langle \tau_1, \ldots \tau_q | \Delta \rangle$$

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Algorithm

Algorithm for Problem 1:

$$\Delta_0 = \Delta$$

$$\Delta_k = \Delta_{k-1} + \sum_{i=1}^{q} [\tau_i, \Delta_{k-1}]$$

$$\text{stop when } \Delta_k = \Delta_{k-1}$$

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