Controllability

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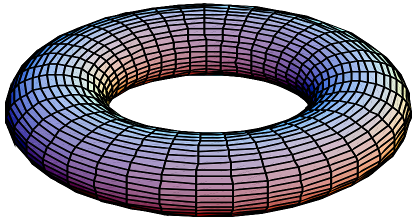

Outline

- Affine Systems
- Controllability
  - Controllability Distributions
  - Controllability Rank Condition
- Examples

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Affine Systems

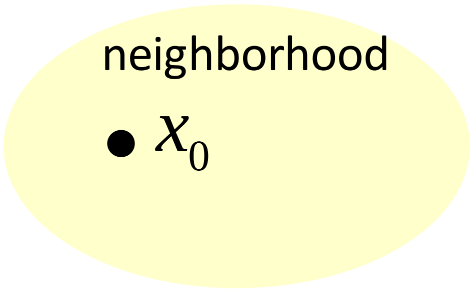
$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

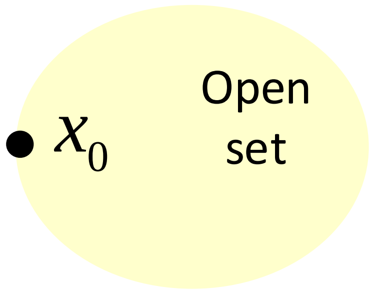
$$y = h(x)$$

$$x \in R^n, y \in R^p, u \in R^m$$

Controllability

- x_f is U-reachable from x_0 if given a neighborhood U of x_0 containing x_f, there exists $t_f>0$ and $u(t)$ on $[0,t_f]$ such that x_0 goes to x_f along a trajectory contained entirely in U.
- The system is <u>locally reachable</u> from x_0 if for each neighborhood U of x_0 the set of states U-reachable from x_0 contains a neighborhood of x_0. If the reachable set contains merely an open set the system is locally <u>weakly</u> reachable from x_0 .
- The system is locally (weakly) controllable if it is locally (weakly) reachable from every initial state.





R. Hermann and A. J. Krener, "Nonlinear Controllability and Observability," *IEEE Transactions on Automatic Control, vol. 22, pp. 728-740, 1977*

Controllability Distributions

$$\Delta_C = \left\langle f, g_1, \ldots, g_m \middle| \operatorname{span}\{ f, g_1, \ldots, g_m \} \right\rangle$$

$$\Delta_{C_O} = \left\langle f, g_1, \ldots, g_m \middle| \operatorname{span}\{ g_1, \ldots, g_m \} \right\rangle$$

Δ_C, Δ_{C_O} satisfy

– $\Delta_{C_O} + \operatorname{span}\{ f \} \subseteq \Delta_C$

– x a regular point of $\Delta_{C_O} + \operatorname{span}\{ f \} \Rightarrow \Delta_{C_O}(x) + \operatorname{span}\{ f(x) \} = \Delta_C(x)$

– if Δ_{C_O} and $\Delta_{C_O} + \operatorname{span}\{ f \}$ are of constant dim, then

$$\dim \Delta_C - \dim \Delta_{C_O} \leq 1$$

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Controllability Rank Condition

Proposition:

A necessary and sufficient condition for the system to be locally weakly controllable is

$$\dim \Delta_C \left(x_0 \right) = n, \; \forall x_0 \in R^n$$

A necessary and sufficient condition for the system to be locally controllable is

$$\dim \Delta_{C_0} \left(x_0 \right) = n, \; \forall x_0 \in R^n$$

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Example: Linear System Controllability

$$\dot{x} = Ax + Bu,\ x \in R^n,\ u \in R^m$$

$$f(x) = Ax,\ g_i(x) = b_i,\ i = 1, \ldots, m$$

$$\left[Ax, b_i\right] = \frac{\partial b_i}{\partial x} Ax - \frac{\partial Ax}{\partial x} b_i = Ab_i$$

$$\left[b_i, b_j\right] = 0, \left[Ax, Ax\right] = 0$$

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Example: Linear System Continued

$$\Delta_0 = \text{span}\{B\}$$

$$\Delta_1 = \text{span}\{B \quad AB\}$$

$$\vdots$$

$$\Delta_k = \text{span}\{B \quad AB \quad \cdots \quad A^{k-1}B\}$$

$$CH - Thm \Rightarrow \Delta_{C_0} = \text{span}\{B \quad AB \quad \cdots \quad A^{n-1}B\}$$

$$\Delta_0 = \Delta$$

$$\Delta_k = \Delta_{k-1} + \sum_{i=1}^{q} [\tau_i, \Delta_{k-1}]$$

$$\text{stop when } \Delta_k = \Delta_{k-1}$$

$$\Delta_0 = \text{span}\{Ax, B\}$$

$$\vdots$$

$$\Delta_C = \text{span}\{Ax, B, AB, \ldots A^{n-1}B\}$$

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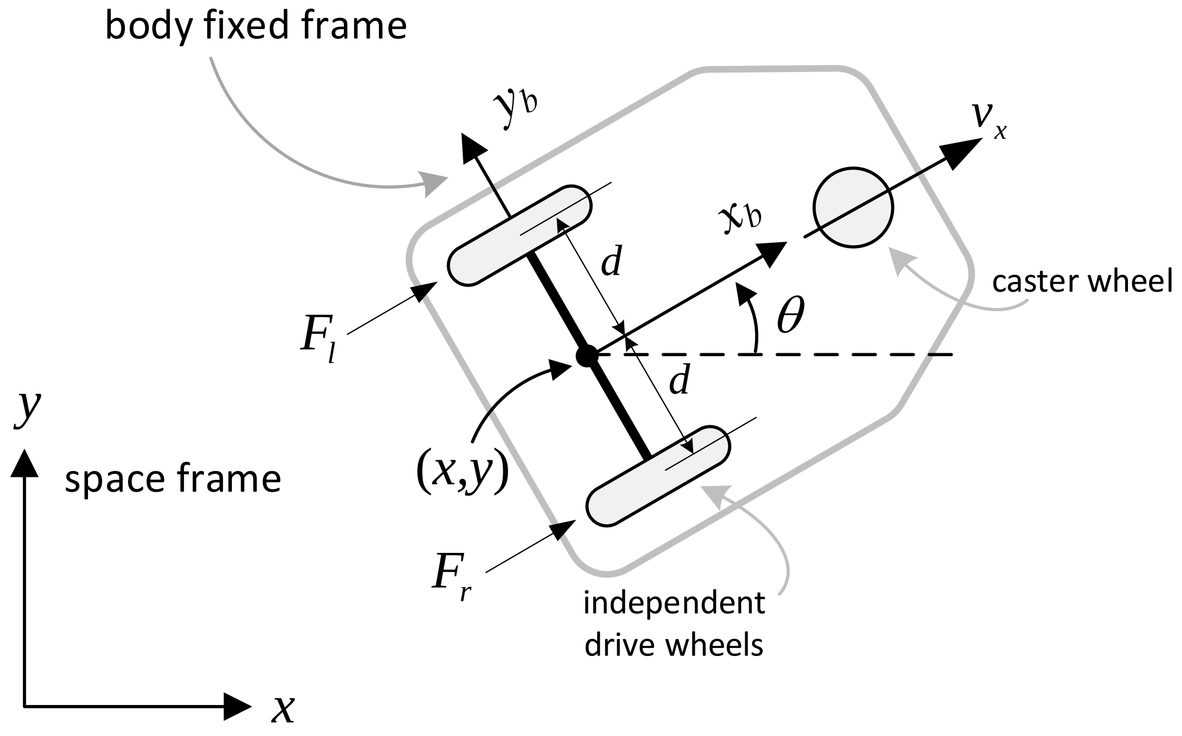
Example: Wheeled Robot

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

drive

rotate



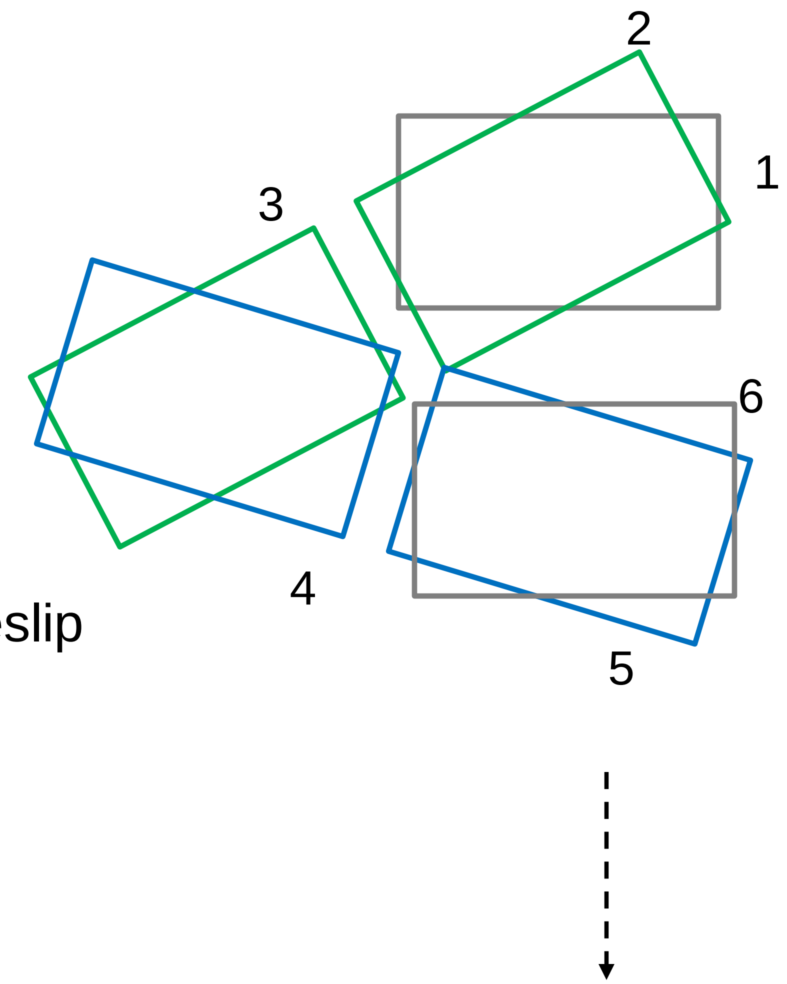
Wheeled Robot 3

$$\Delta_0 = \text{span}\{\text{drive}, \text{rotate}\} = \left\{ \begin{bmatrix} 1 \\ \tan\theta \\ \theta \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

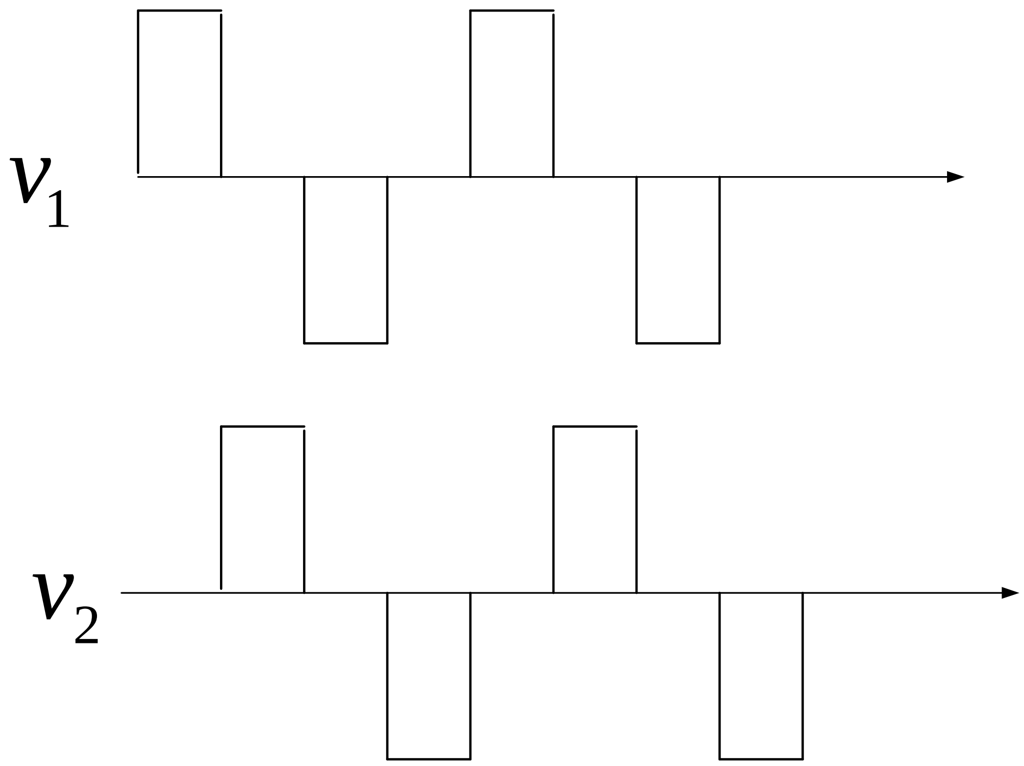
$$\left[\begin{bmatrix} 1 \\ \tan\theta \\ \theta \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 \\ \sec^2\theta \\ 0 \end{bmatrix}$$

sideslip

$$\text{span}\left\{ \text{drive}, \text{rotate}, \begin{bmatrix} 0 \\ \sec^2\theta \\ 0 \end{bmatrix} \right\} = I_3$$



Implementing Lie Bracket



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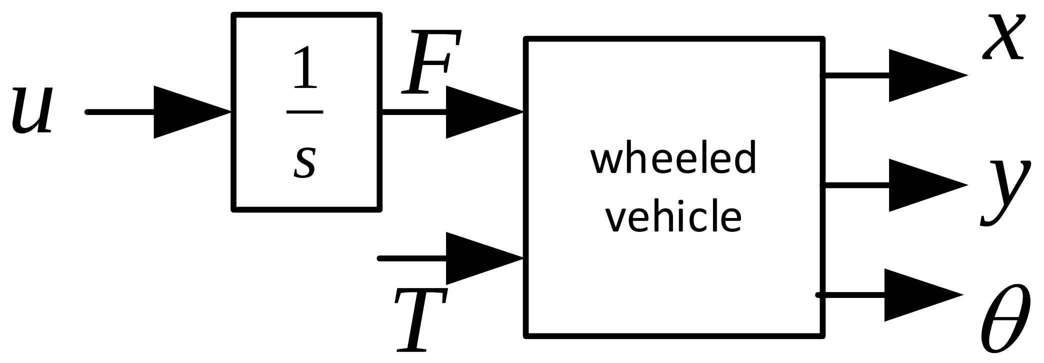
Example: Extended Wheeled Robot

The extended system includes dynamics and $z = F, \dot{z} = u$:

$$\frac{d}{dt}\begin{bmatrix}\theta \\ x \\ y \\ v_x \\ \omega \\ z\end{bmatrix} = \begin{bmatrix}\omega \\ v_x \cos\theta \\ v_x \sin\theta \\ z \\ 0 \\ 0\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}u \\ T\end{bmatrix}$$



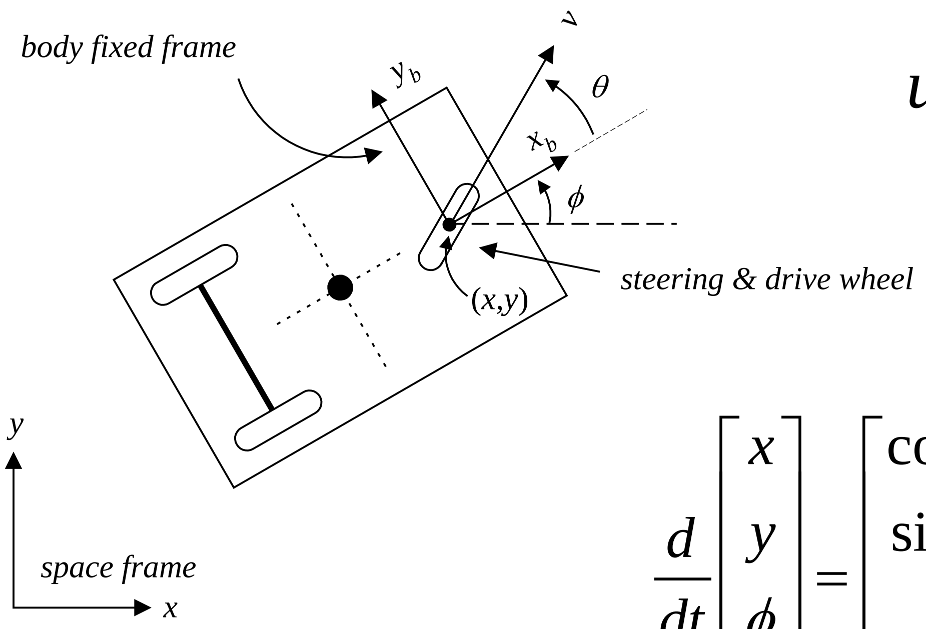
Example: Extended Wheeled Robot - 2

$$\Delta_{C_0} = \text{span}\left\{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix},\ldots,\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right\}, \text{rank } 6 \Rightarrow \text{controllable at generic state}$$

$$\Delta_{C_0}\Big|_{x=0} = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\\tan\theta\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\right\}, \text{rank } 5 \Rightarrow \text{uncontrollable at } x = 0$$

Actually, this is true not only at the origin,
but at any point with v_x and ω both zero.
For generic points with only $v_x = 0$,
the system is controllable.

Example: Parking



$$u_1 = v, u_2 = \omega$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\phi+\theta) & 0 \\ \sin(\phi+\theta) & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

drive

steer

Parking, Continued

$$\Delta_C = \left\langle f, g_1, \ldots, g_m \middle| \text{span}\left\{ f, g_1, \ldots, g_m \right\} \right\rangle$$

$$\Delta_{C_O} = \left\langle f, g_1, \ldots, g_m \middle| \text{span}\left\{ g_1, \ldots, g_m \right\} \right\rangle$$

$$\Delta_C = \Delta_{C_O} = \left\langle \begin{bmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| \text{span}\left\{ \begin{bmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

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Example: Parking, new directions from Lie bracket

$$wriggle = [steer, drive] = \begin{bmatrix} -\sin(\theta+\phi) \\ \cos(\theta+\phi) \\ \cos\theta \\ 0 \end{bmatrix}$$

$$slide = [wriggle, drive] = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \\ 0 \end{bmatrix}$$

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Parking, Continued

$$\operatorname{span}\left\{\begin{bmatrix}\cos(\phi+\theta)\\ \sin(\phi+\theta)\\ \sin(\theta)\\ 0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\begin{bmatrix}-\sin(\theta+\phi)\\ \cos(\theta+\phi)\\ \cos\theta\\ 0\end{bmatrix},\begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\\ 0\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

Recall Lie Bracket Interpretation as Commutator of Flows

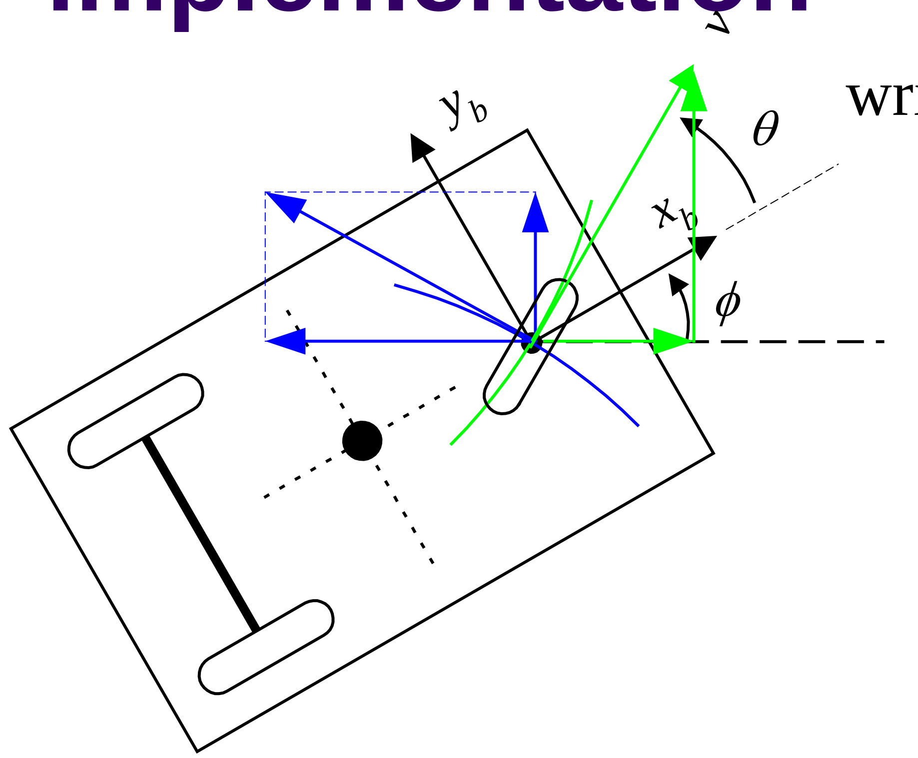
Let us consider the Lie bracket as a commutator of flows. Beginning at point x in M follow the flow generated by v for an infinitesimal time which we take as $\sqrt{\varepsilon}$ for convenience. This takes us to point

$$y = \exp(\sqrt{\varepsilon}\ \mathbf{v})x$$

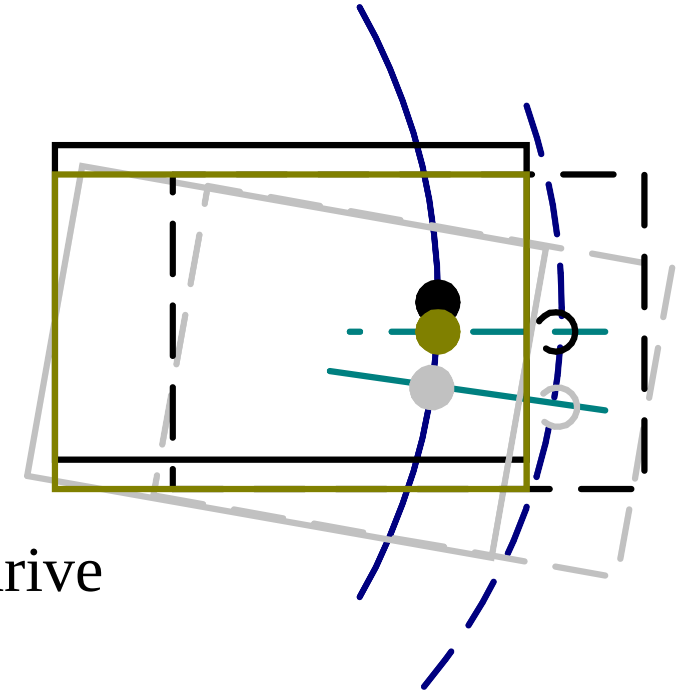
Then follow **w** for the same length of time, then **-v**, then **-w**. This brings us to a point ψ given by

$$\psi(\varepsilon, x) = e^{-\sqrt{\varepsilon}\mathbf{w}}\, e^{-\sqrt{\varepsilon}\mathbf{v}}\, e^{\sqrt{\varepsilon}\mathbf{w}}\, e^{\sqrt{\varepsilon}\mathbf{v}} x$$

Example: Parking, implementation



wriggle=steer+drive-steer-drive



slide=wriggle+drive-wriggle-drive

More Controllability Distributions

$$\Delta_L = \text{span}\left\{ f, ad_f^k g_i, 1 \le i \le m, 0 \le k \le n-1 \right\}$$

$$\Delta_{L_0} = \text{span}\left\{ ad_f^k g_i, 1 \le i \le m, 0 \le k \le n-1 \right\}$$

$$ad_v^0(w) = w,\ ad_v^k = \left[v, ad_v^{k-1}(w) \right]$$

What is missing in these earlier attempts to obtain a controllability determination is any Lie Brackets between fields g_i, g_j.

They are sufficient but not necessary conditions for controllability.

$$\begin{array}{ccccccc} \textit{weak local controllability} & \Leftrightarrow & \Delta_C = n & \Leftarrow & \Delta_L = n \\ \Uparrow & & \Uparrow & & \Uparrow \\ \textit{local controllability} & \Leftrightarrow & \Delta_{C_0} = n & \Leftarrow & \Delta_{L_0} = n \end{array}$$

Example: Linear Systems Revisited

$$f(x) = Ax, G(x) = B$$

$$\Delta_C = \Delta_L = \text{span}\left\{Ax, B, AB, \ldots, A^{n-1}B\right\}$$

$$\Delta_{C_0} = \Delta_{L_0} = \text{span}\left\{B, AB, \ldots, A^{n-1}B\right\}$$

Controllability Hierarchy

$$\Delta_C = \left\langle f, g_1, \ldots, g_m \middle| \text{span}\{ f, g_1, \ldots, g_m \} \right\rangle$$

$$\Delta_{C_O} = \left\langle f, g_1, \ldots, g_m \middle| \text{span}\{ g_1, \ldots, g_m \} \right\rangle$$

$$\Delta_L = \text{span}\left\{ f, ad_f^k g_i, 1 \le i \le m, 0 \le k \le n-1 \right\}$$

$$\Delta_{L_0} = \text{span}\left\{ ad_f^k g_i, 1 \le i \le m, 0 \le k \le n-1 \right\}$$

$$\begin{array}{ccccc}
\textit{weak local controllability} & \Leftarrow & \dim \Delta_C(x_0) = n & \Leftarrow & \dim \Delta_L(x_0) = n \\
\Uparrow & & \Uparrow & & \Uparrow \\
\textit{local controllability} & \Leftarrow & \dim \Delta_{C_0}(x_0) = n & \Leftarrow & \dim \Delta_{L_0}(x_0) = n \\
\Uparrow & & & & \Updownarrow \\
\textit{linear controllability} & & \Leftrightarrow & & \dim\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n
\end{array}$$

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