# Observer Design

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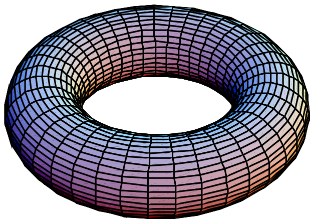

# Outline

- Problem definition
- Observability hierarchy
  - Singular inputs
- Local exponential observers
  - Constant Gain Observers
- Global observers
  - High Gain Observer
  - Detectable Systems
  - Compensating for uncertain nonlinearity
- Variable structure
- Linearization up to output injection
- Normal form observers

# Problem Definition

Given a system:

$$\dot{x} = f(x) + G(x)u = f_u(x)$$

$$y = h(x)$$

$$x \in R^n, u \in R^m, y \in R^p$$

$$f(x_0) = 0, h(x_0) = 0$$

Construct an observer, i.e., an estimator:

$$\hat{x}\left(t \middle| y(\tau), u(\tau), \tau \in [0, t]\right)$$

Such that

$$\left\| x(t) - \hat{x}(t) \right\| \to 0 \text{ as } t \to \infty$$

# Observability

- Consider an open set $U$ in $R^n$. $x_1,x_2$ are $U$-distinguishable if there exists a control u(t) whose trajectories from both $x_1,x_2$ remain in $U$ such that $y(t;x_1,u) \neq y(t;x_2,u)$. Otherwise they are $U$-indistinguishable.
- The system is strongly locally observable at $x_0$ if for every nbhd $U$ of $x_0$ every state in $U$ other than $x_0$ is $U$-distinguishable from $x_0$. It is locally observable at $x_0$ if there exists a nbhd $W$ of $x_0$ such that for every nbhd $U$ *of* $x_0$ contained in $W$ every state in $U$ other than $x_0$ is $U$-distinguishable from $x_0$.
- The system is (strongly) locally observable if it is (strongly) locally observable at $x_0$ for every $x_0$ in $R^n$.

Clearly, strong local observability $\Rightarrow$ local observability.

Local observability only requires that states sufficiently close to $x_0$ are distinguishable from $x_0$.

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# Observability Codistributions

$$\Omega_O = \left\langle f, g_1, \ldots, g_m \middle| \mathrm{span}\left\{dh_1, \ldots, dh_p\right\}\right\rangle$$

$$\Delta_O = \Omega_O^{\perp}$$

$$\Omega_L = span\left\{L_f^k\left(dh_i\right), 1 \le i \le p, 0 \le k \le n-1\right\}$$

The distribution $\Delta_O$ is invariant wrt $f, g_1, \ldots, g_m$ and and it is contained in the kernel of span$\{dh_1, \ldots, dh_p\}$. If it is nonsingular, it is also involutive.

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# Observability Rank Condition

**Proposition:** If $\Omega_O$ (equivalently, $\Delta_O$) is of constant dimension on some open set $U$, then the system is locally observable on $U$ if and only if $\dim \Omega_O = n$, or equivalently, $\dim \Delta_O = 0$.

# Example: Linear System Observability

$$\dot{x} = Ax + Bu,\ y = Cx$$

$$f(x) = Ax,\ g_i(x) = b_i,\ dh_j = c_j \Rightarrow L_{Ax}c_j = c_j A,\ L_{b_i}c_j = 0$$

$$\Omega_0 = \operatorname{span}\{C\},\ \Omega_1 = \operatorname{span}\begin{Bmatrix} C \\ CA \end{Bmatrix},\ldots,\Omega_k = \operatorname{span}\begin{Bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{Bmatrix}$$

$$CH\ Thm \Rightarrow \operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

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# Example: Role of Input

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1 \\ 0 \end{bmatrix} u, \quad y = x_2$$

- The linearized system is not observable,
- The system with g(x)=0, yields

$$\Omega_O = \text{span}\left\{ \begin{matrix} [0 & 0 & 1] \\ [0 & 1 & 0] \end{matrix} \right\}$$

- On the other hand, for this system

$$\Omega_O = \text{span}\left\{ \begin{matrix} [1 & 0 & 0] \\ [0 & 1 & 0] \\ [0 & 0 & 1] \end{matrix} \right\}$$

# Observability Hierarchy

$$
\begin{array}{ccc}
\dim \Omega_O(x_0) = n & \Rightarrow & \textit{locally observable} \\
\Uparrow & & \Uparrow \\
\dim \Omega_L(x_0) = n & \Rightarrow & \textit{zero input observable} \\
\Updownarrow & & \Uparrow \\
\dim \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n & \Leftrightarrow & \textit{linearly observable}
\end{array}
$$

$$\Omega_O = \left\langle f, g_1, \ldots, g_m \middle| \operatorname{span}\left\{ dh_1, \ldots, dh_p \right\} \right\rangle \qquad \Omega_L = \left\{ L_f^k \left( dh_i \right) \middle| 1 \le i \le p, 0 \le k \le n-1 \right\}$$

# Observers that Mirror System Dynamics

$$\text{system: } \dot{x} = f\left(x,u\right),\ y = h\left(x\right)$$

$$\text{observer: } \dot{\hat{x}} = f\left(\hat{x},u\right) + \kappa\left(\hat{x}, y - h(\hat{x})\right)$$

$$\text{error: } e \triangleq x - \hat{x}$$

$$\dot{e} = f\left(x,u\right) - f\left(x-e,u\right) + \kappa\left(x-e, h(x) - h(x-e)\right)$$

$$e = 0 \Rightarrow \dot{e} = 0\ \forall u(t), x(t)$$

$$\textit{only if}\ \ \kappa\left(x,0\right) \equiv 0$$

$$\kappa\left(\cdot,\cdot\right) \text{ drives } e(t) \to 0$$

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# Exponental Detectability

A system is exponentially detectable at $(x^*,u^*)$ if there exists a function $\gamma(x,y)$

- $\gamma(x^*, y^*) = 0$
- $\gamma\left(\xi, h(\xi)\right) = f\left(\xi, u^*\right)$
- $\xi = x^*$ is an exp stable e.p. of $\dot{\xi} = \gamma\left(\xi, y^*\right)$

Exponential detectability

$\Downarrow$

$$\dot{\hat{x}} = f\left(\hat{x},u\right) - f\left(\hat{x},u^*\right) - \gamma\left(\hat{x}, y\right)$$

is a local observer

$$\dot{\hat{x}} = f\left(\hat{x},u\right) + \kappa\left(\hat{x}, y - h(\hat{x})\right)$$

$$\gamma\left(\xi,y\right) \Leftrightarrow f\left(\xi,u^*\right) + \kappa\left(\xi, y - h(\xi)\right)$$

# Constant Gain Observer

$$\dot{x} = f(x,u) = Ax + Bu + h.o.t.$$

$$y = Cx + h.o.t.$$

$(A, C)$ detectable

choose: $\kappa\left(\hat{x}, y\right) = L\left(h(\hat{x}) - y\right)$, $\operatorname{Re}\lambda\left(A + LC\right) < 0$

$$L = -SC^T V^{-1}$$

$$SA^T + AS - SC^T V^{-1} CS = -W,\ S \geq 0$$

# Global High Gain Observer

$$f(x,u) = F(x) + G(x)u$$

- locally uniformly observable for all inputs $\Rightarrow \left(A, C\right)$ observable
- $g_i(x)$ globally Lipschitz

$\Rightarrow$ for inputs $u$ uniformly bounded, & $\theta$ sufficiently large

$\dot{\hat{x}} = F(\hat{x}) + G(\hat{x})u + L\left(C\hat{x} - y\right)$ is an observer, where

$$L = -SC^T V^{-1}$$

$$\left(A + \theta I\right)S + S\left(A + \theta I\right)^T - SC^T V^{-1} CS = -W, \; S \geq 0$$

Gauthier, Hammouri, Othman 1992

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# Global Observer for Detectable Systems

$$f(x,u) = Ax + f^2(x,u),\ y = Cx$$

- $(A,C)$ detectable $\Rightarrow \exists L,\ P>0,\ Q>0$ *such that* $(A+LC)P + P(A+LC)^T = -Q$
- $f^2(x,u)$ globally $\gamma-$Lipschitz in $x$ uniformly $\forall u$
- $\dfrac{\lambda_{\min} Q}{2\lambda_{\max} P} > \gamma$

$\Rightarrow \dot{\hat{x}} = A\hat{x} + f^2(\hat{x},u) + L(C\hat{x} - y)$ is an observer

Thau, 1973
Raghavan & Hedrick, 1994
Rajamani, 1998

$$L = -SC^T V^{-1}$$

$$AS + SA^T - SC^T V^{-1} CS = -W,\ S \geq 0$$

$$\left(A - SC^T V^{-1} C\right) S + S\left(A - SC^T V^{-1} C\right)^T = -W - SC^T V^{-1} CS$$

$$S \to P,\ W + SC^T V^{-1} CS \to Q$$

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# Global Robust Observer

$f(x,u) = Ax + f^2(x,u),\ y = Cx$

- $(A,C)$ detectable $\Rightarrow \exists L,\ P>0,\ Q>0\ \textit{such that}\ (A+LC)P + P(A+LC)^T = -Q$
- $\exists \eta : R^n \times R^m \to R^p\ \ \textit{such that}\ \ f^2(x,u) = P^{-1}C^T\eta(x,u)$
- $\|\eta(x,u)\| < \rho(t,u)\ \forall x,u$

$\Rightarrow \dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + S(C\hat{x} - y, \rho)$ is an observer

$$S(\xi,\rho) = P^{-1}C^T\xi\rho^2 / \left\|\xi\rho + \varepsilon e^{-\beta t}\right\|$$

Moreover, $\|\hat{x} - x\|^2 \le ae^{-bt},\ b = \lambda_{\min}(Q)/\lambda_{\max}(P)$

Walcott & Zac, 1987    Dawson, Qu, Carroll, 1992

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# Observer Design via Linearization to Output Injection

Krener, Isidori, Respondek

$$\Sigma : \begin{array}{l} \dot{x} = f(x) \\ y = h(x) \end{array} \Rightarrow \Sigma_L : \begin{array}{l} \dot{z} = Az + \varphi(y) \\ y = Cz \end{array}$$

$$\dot{\hat{z}} = A\hat{z} + \varphi(y) + L\left(C\hat{z} - y\right)$$
$$\dot{e} = \left(A + LC\right)e, \; e = z - \hat{z}$$

Hammouri, Gauthier, Kinnaert

$$\Sigma : \begin{array}{l} \dot{x} = f(x) + G(x)u \\ y = h(x) \end{array} \Rightarrow \Sigma_L : \begin{array}{l} \dot{z} = A\left(u(t)\right)z + \varphi\left(y, u(t)\right) \\ y = Cz \end{array}$$

$$\dot{\hat{z}} = A\left(u(t)\right)\hat{z} + \varphi\left(y, u(t)\right) - P(t)C^T\left(C\hat{z} - y\right)$$
$$\dot{P} = PA^T\left(u(t)\right) + A\left(u(t)\right)P - PC^TCP + Q$$

$$\dot{e} = \left[A\left(u(t)\right) + P(t)C^TC\right]e$$

# Example: Krener & Respondek

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^2 + x_1^2 x_2 \\ -x_2 x_3 / \left(1 + x_1\right) \end{bmatrix}, \quad y = \begin{bmatrix} x_1 \\ x_3 + x_1 x_3 \end{bmatrix}$$

$$z_1 = x_1,\ z_2 = \left(1 + x_1\right) x_3,\ z_3 = \left(x_1^3 - 3x_2\right)/3$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} y_1^3/3 \\ 0 \\ -y_1^2 \end{bmatrix}, \quad y = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

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# Example: Krener & Respondek

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 + \left(1 + e^{x_1}\right) u \\ 3x_1^2 x_2^2 + x_1^3 x_3 + \left(1 + x_1 + x_2\right) u \end{bmatrix}, \quad y = x_1$$

$$z_1 = x_1,\ z_2 = \left(x_1^4 - 4x_2\right)/2,\ z_3 = -x_1^3 x_2 + x_3$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 0 \\ 0 & 0 & -2 \\ 0 & -\frac{1}{2}u & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} y^4 \\ -2\left(1 + e^y\right) u \\ \left(1 + y - \left(1 + e^y\right) y^3 + y^4/4\right) u \end{bmatrix}$$

# Example: Hammouri & Kinnaert Modified

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} e^{x_1+x_2} - 1 + u x_2^2 \\ -e^{x_1+x_2} + 1 + u\left(e^{x_3-x_2} - e^{-x_1-x_2} - x_1^2\right) \\ -e^{x_1+x_2} + 1 + x_1^3 - u x_1^2 \\ x_5 \\ x_1 \end{bmatrix} \qquad y = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$$

$$z_1 = x_1,\ z_2 = x_4,\ z_3 = -e^{x_1+x_2},\ z_4 = -x_5,\ z_5 = e^{x_1+x_3}$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} -z_3 - 1 + u z_1^2 \\ -z_4 \\ u\left(1 - z_5\right) \\ -z_1 \\ z_1^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} + \begin{bmatrix} -1 + y_1^2 u \\ y_2 \\ u \\ -y_1 \\ y_1^3 \end{bmatrix}$$

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# Xia & Zeitz: Example 3

This system is locally observable but not zero input observable

$$\dot{x}_1 = x_2^3 \qquad\qquad \dot{z}_1 = -z_2$$

$$\dot{x}_2 = x_2 u \quad \Leftrightarrow \quad \dot{z}_2 = 3z_2 u$$

$$y = x_1 \qquad\qquad y = z_1$$

smooth

continuous

$$z_1 = x_1 \qquad\qquad x_1 = z_1$$

$$z_2 = -x_2^3 \quad \Leftrightarrow \quad x_2 = -z_2^{1/3}$$