Control Design via Feedback Linearization

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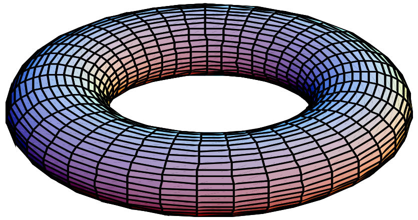

Outline

- Definitions & Notation
  - Lie Derivative, Exponential Map,Lie Bracket
- Input-output feedback linearization
  - SISO, MIMO
  - Relative degree & zero dynamics
- Exact (state) feedback linearization
- Dynamic extension
- Regulation of feedback linearizable systems

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Setup

$$\dot{x} = f(x) + G(x)u$$
$$y = h(x)$$

$$x \in R^n, u \in R^m, y \in R^p$$
$$f(x), G(x) = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix}, h(x) \text{ smooth}$$

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I-O Linearization: SISO case

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = L_f h(x) + L_g h(x)u$$

$$\text{if } L_g h(x) \neq 0 \text{ stop, else}$$

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x)u$$

$$\text{if } L_g L_f^{k-1} h(x) \neq 0 \text{ stop, else continue until}$$

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)u \text{ with } L_g L_f^{r-1} h(x) \neq 0$$

$$\text{set } u = \frac{1}{L_g L_f^{r-1} h(x)}\left(-L_f^r h(x) + v\right) \Rightarrow \boxed{y^{(r)} = v}$$

Relative Degree

Definition: Let U be a neighborhood of x_0 and suppose there is a finite integer r such that

$$L_g L_f^k h(x) = 0, \forall x \in U, k = 0, \ldots, r-2, L_g L_f^{r-1} h(x) \neq 0$$

Then r is the relative degree. If the sequence does not terminate in finite steps, the system relative degree is

$$r = \infty$$

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State Transformation

- r finite $\Rightarrow r \leq n$

define: $z_i(x) := L_f^{i-1} h(x),\ i = 1, \ldots, r$

- $z_i(x),\ i = 1, \ldots, r$ are independent functions

note: $z_1 = y,\ z_2 = \dot{y}, \ldots, z_r = y^{(r-1)}$

- complete the transformation by adding functions

$define: \xi_i(x),\ i = 1, \ldots, n - r$ independent of $z(x)$

$$\Downarrow$$

$$x \mapsto (\xi, z)$$

Normal Form

$$\dot{\xi} = F(\xi, z, u) = F(\xi, z) \text{ special choice of } \xi$$

$$\dot{z} = Az + b\left[\alpha\left(x(\xi, z)\right) + \rho\left(x(\xi, z)\right)u\right]$$

$$y = cz$$

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\alpha(x) = L_f^r h(x), \quad \rho(x) = L_g L_f^{r-1} h(x)$$

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Feedback Linearization & Zero Dynamics

$$u = -\rho^{-1}(x)\left\{\alpha(x) + v\right\}$$

$$\Downarrow$$

$\dot{\xi} = F(\xi, z)$ internal dynamics

$\dot{z} = Az + bv$

$y = cz$

I-O linearized dynamics

$\dot{\xi} = F(\xi, 0)$ zero dynamics

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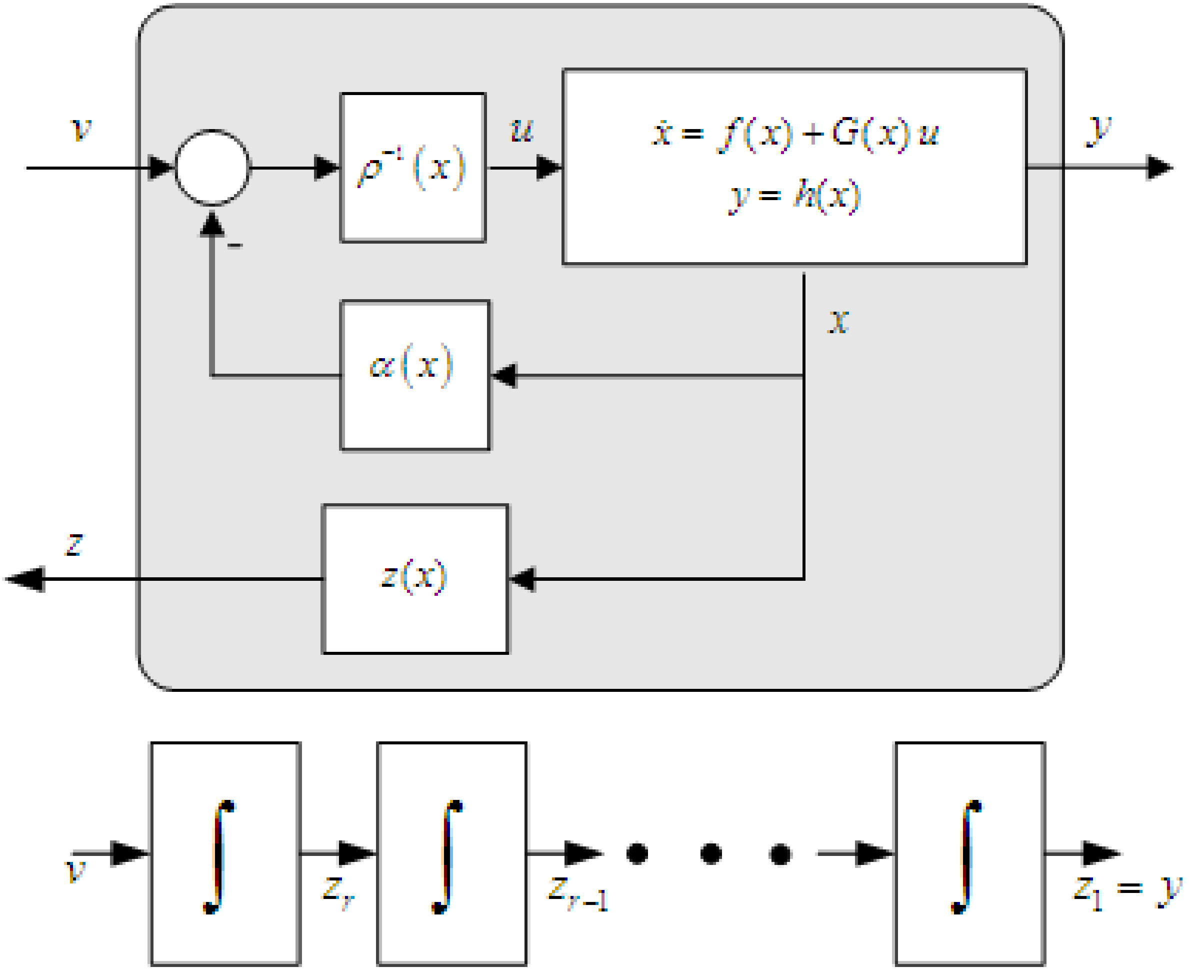
Chain of Integrators



Example 1

Design an I/O linearizing control system with $y \to 0$

$$\dot{x}_1 = x_2 + x_1^3 + u$$

$$\dot{x}_2 = -u$$

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2 + x_1^3 + u$$

$$\Rightarrow r = 1, \alpha(x) = x_2 + x_1^3, \beta(x) = 1, u = -x_2 - x_1^3 + v$$

$$z_1(x) = x_1$$

we need one more function to complete the transformation

choose $\xi_1(x) = x_2$

$\boxed{\dot{z}_1 = v}$ I/O linearized dynamics

$\dot{\xi}_1 = -u = x_2 + x_1^3 - v \Rightarrow \boxed{\dot{\xi}_1 = \xi_1 + z_1^3 - v}$ internal dynamics

$\boxed{\dot{\xi}_1 = \xi_1}$ zero dynamics (unstable!)

Example 1, Cont'd

What can be done in the case of unstable zero dynamics?

The conventional way is to try and redefine the output function. Two ways of approaching this are:

- Try to find a new $h(x)$ such that the zero dynamics are stable.
- Try to find a new $h(x)$ such that the system is full state linearizable, i.e., $r = n$.

$$y = h(x) = x_1 + x_2$$

$$\dot{y} = x_2 + x_1^3, \quad \ddot{y} = 3x_1^2\left(x_2 + x_1^3\right) + \left(3x_1^2 - 1\right)u$$

$$\Rightarrow r = 2, \alpha = 3x_1^2\left(x_2 + x_1^3\right), \beta = \left(3x_1^2 - 1\right)$$

problem if $x_1 = \pm\sqrt{1/3}$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2+2x_3 \\ 1 \\ 0 \end{bmatrix} u, \quad y = x_4$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_1^2 + x_2 \\ x_1 \\ \frac{1}{2}\left(-x_2 + 2x_3 + x_3^2\right) \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_2 - z_3^2 \\ -1 - \sqrt{1 + z_2 - z_3^2 - z_4} \\ z_1 \end{bmatrix}$$

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Example, Cont'd

Normal form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \hline \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 + 2z_2 z_3^2 - 4z_3^4 \\ \hline z_3\left(z_2 - 2z_3^2\right) \\ -1 - z_2 - z_3/2 + z_3^2 - \sqrt{1 + z_2 - z_3^2 + 2z_4} - 2z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -2\sqrt{1 + z_2 - z_3^2 - 2z_4} \\ \hline 0 \\ 0 \end{bmatrix} u$$

Zero dynamics

$$\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -2z_3^3 \\ -1 - z_3/2 + z_3^2 - \sqrt{1 - z_3^2 + 2z_4} - 2z_4 \end{bmatrix}$$

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I-O Linearization: MIMO case

$$\text{assume } p = m$$

$$r_i = \inf\left\{k \,\middle|\, L_{g_j} L_f^{k-1} h_i(x) \neq 0 \;\; \textit{for at least one } j\right\}$$

$$\alpha_i(x) = L_f^{r_i} h_i(x),\; i = 1, \ldots, m$$

$$\rho_{ij} = L_{g_j} L_f^{r_i} h_i(x),\; i, j = 1, \ldots, m$$

Definition: if $\det\{\rho(x)\} \neq 0$, $\mathbf{r} = \begin{bmatrix} r_1 & \cdots & r_m \end{bmatrix}$ is the vector relative degree.

MIMO State Transformation

state transformation $, x \to (\xi, z), \xi \in R^{n-r}, z \in R^{r}, r = r_1 + \cdots + r_m$

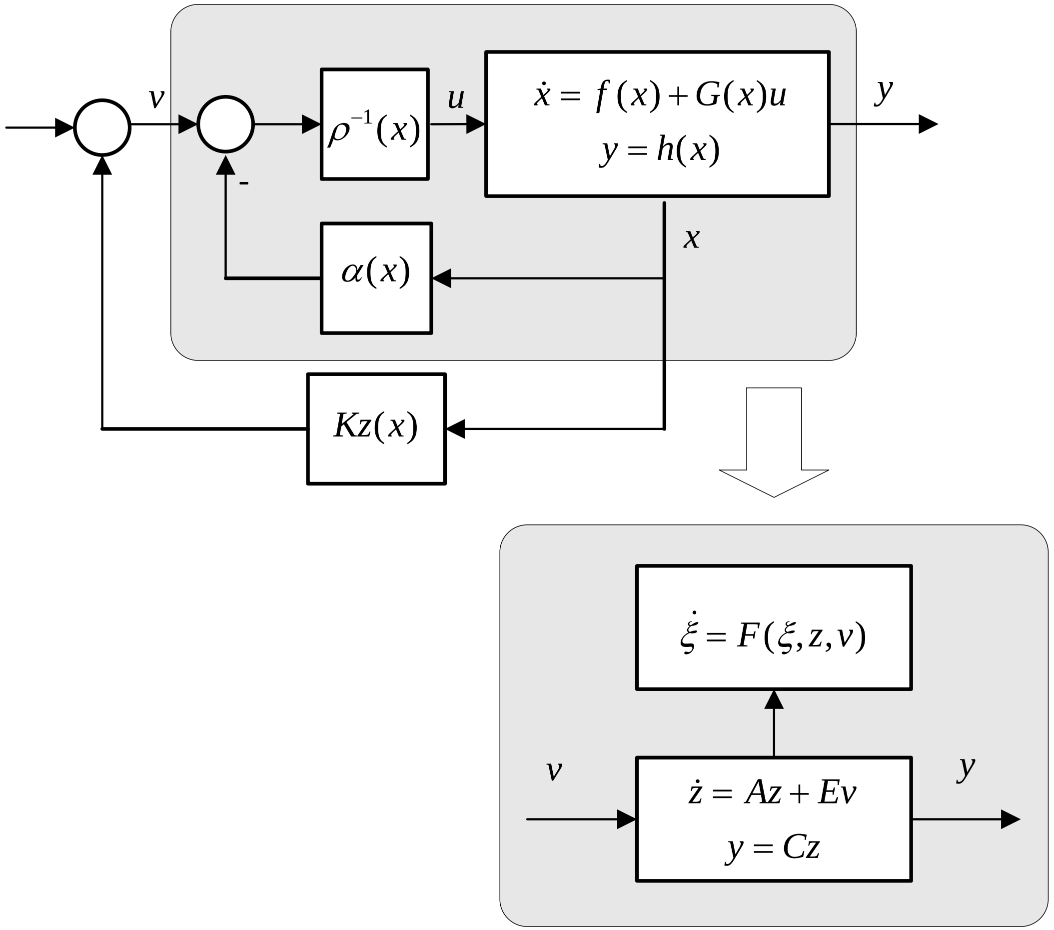
$$z(x) = \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_m \end{bmatrix} := \begin{bmatrix} h_1 \\ \vdots \\ L_f^{r_1-1}(h_1) \\ \vdots \\ \vdots \\ h_m \\ \vdots \\ L_f^{r_m-1}(h_m) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_1^{r_1-1} \\ \vdots \\ \vdots \\ y_m \\ \vdots \\ y_m^{r_m-1} \end{bmatrix}$$

$$\dot{\xi} = F(\xi, z, u)$$

$$\dot{z} = Az + E\left[\alpha\left(x(\xi, z)\right) + \rho\left(x(\xi, z)\right)u\right]$$

$$y = Cz$$

I-O Linearization Summary



Computational Tools

Function Name	
VectorRelativeOrder	computes the relative degree vector
DecouplingMatrix	computes the decoupling matrix, $\rho(x)$
IOLinearize	computes the linearizing control, $u = \rho^{-1}(x)\{-\alpha(x) + v\}$
NormalCoordinates	computes the partial state transformation, $z(x)$
LocalZeroDynamics	computes the local form of the zero dynamics $F(\xi, 0)$, near x_0

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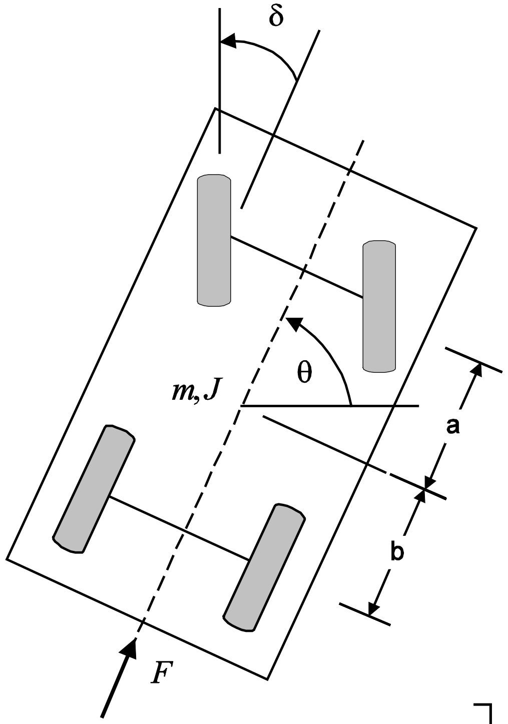
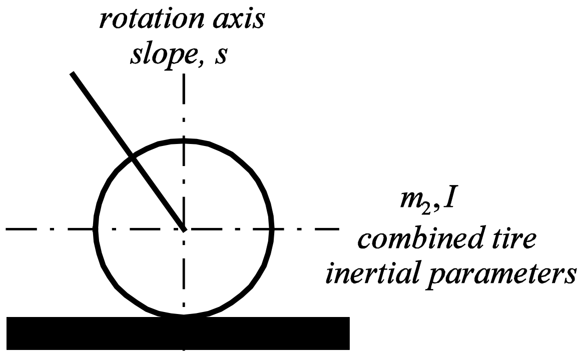
Example: Simple Vehicle





Assumptions:

$m_2 = 0,\ s = 0$

$$\frac{d}{dt}\begin{bmatrix}\omega_\theta \\ v_x \\ v_y \\ \omega_\delta \\ \delta\end{bmatrix} = \begin{bmatrix} -\frac{\kappa}{J_{zz}v_x}\left(\left(a^2+b^2\right)\omega_\theta + (a-b)v_y - av_x\delta\right) \\ -\frac{1}{m_1}\left(m_1\omega_\theta v_y - \kappa\left(a\omega_\theta + v_y\right)\frac{\delta}{v_x}\right) \\ \omega_\theta v_x - \kappa\left(\left(a-b\right)\omega_\theta + 2v_y - v_x\delta\right)\frac{1}{m_1 v_x} \\ -\frac{\kappa}{J_{zz}v_x}\left(-\left(a^2+b^2\right)\omega_\theta + (a-b)v_y - av_x\delta\right) \\ \omega_\delta \end{bmatrix} + \begin{bmatrix} -\frac{1}{J_{zz}} & 0 \\ 0 & \frac{1}{m_1} \\ 0 & 0 \\ \frac{1}{I_{zz}}+\frac{1}{J_{zz}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ F \end{bmatrix}$$

Example, Cont'd

Objective: Steer vehicle along a circular path of constant radius R and constant speed V_d. Equivalently, specify constant angular velocity $\omega_d = V_d / R$ and define two outputs

$$y_1 = v_x^2 + v_y^2 - V_d^2$$

$$y_2 = \omega_\theta - \omega_d$$

Example, Cont'd

```
In[20]:= {rho, alpha, ro, control} =
           IOLinearize[f27, g27, h27, var27];
         FullSimplify[alpha] // MatrixForm
         FullSimplify[rho] // MatrixForm

         Computing Decoupling Matrix

         Computing linearizing/decoupling control
```

Out[21]//MatrixForm=

$$\begin{pmatrix} \frac{2\,(a\,x1\,(-x3+x2\,x5)+x3\,(b\,x1-2\,x3+2\,x2\,x5))\,x}{m1\,x2} \\ \frac{\left(-a^2\,x1+b\,(-b\,x1+x3)+a\,(-x3+x2\,x5)\right)\,x}{Jzz\,x2} \end{pmatrix}$$

Out[22]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{2\,x2}{m1} \\ -\frac{1}{Jzz} & 0 \end{pmatrix}$$

Example: Dynamic Extension

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rho = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix} \quad \text{singular!}$$

$$\text{define: } \dot{u}_1 = v \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{u}_1 \end{bmatrix} = \begin{bmatrix} u_1 \cos x_3 \\ u_1 \sin x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ u_2 \end{bmatrix}$$

$$\rho = \begin{bmatrix} \cos x_3 & -u_1 \sin x_3 \\ \sin x_3 & u_1 \cos x_3 \end{bmatrix}$$

nonsingular!

Dynamic Extension Algorithm

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad y = h(x)$$

1) Compute $\rho(x)$. If rank$\rho = m$ Stop!

2) If rank$\rho = s < m$, perform elementary column operations to make the first s columns independent and the last $m - s$ columns zero. $\rho_1(x) = \rho(x)E(x)$, $E(x)$ square, nonsingular.

3) Suppose there are q columns each having two or more elements that are not identically zero. If $q = 0$, Stop! The process fails.

4) If $q \neq 0$, define the column index set $\alpha = \{i_1, \ldots, i_q\}$ and let $\bar{\alpha}$ denote its complement in $\{1, \ldots, m\}$. Put and integrator in series with the q corresponding controls to obtain a new augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{u}_\alpha \end{bmatrix} = \begin{bmatrix} f(x) + \sum_{i \in \alpha} g_i(x) u_i \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{i \in \bar{\alpha}} g_i(x) u_i \\ v_\alpha \end{bmatrix}$$

5) Go to step 1 and repeat the process with the new augmented system.

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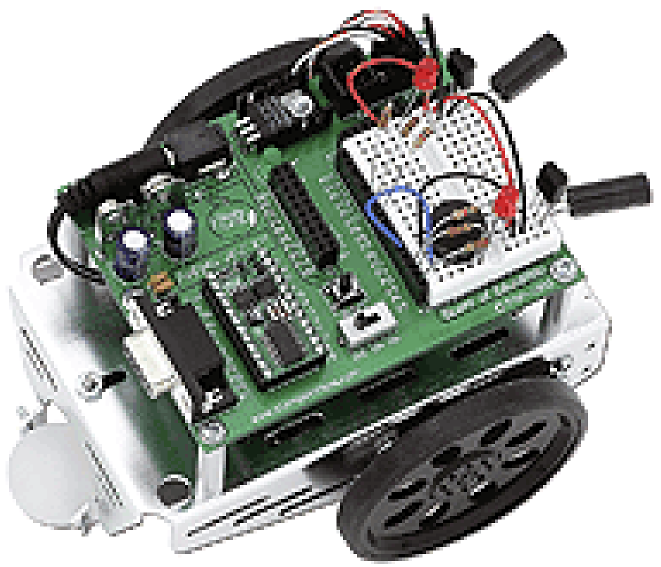
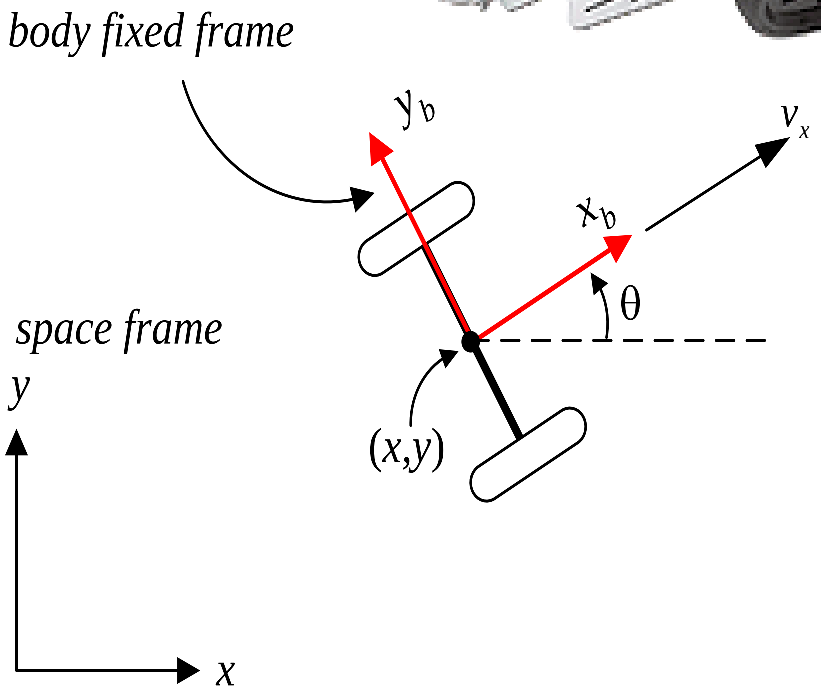
Example: Wheeled Robot

$$\dot{x} = v_x \cos\theta$$
$$\dot{y} = v_x \sin\theta$$
$$\dot{\theta} = \omega$$
$$M\dot{v}_x = F$$
$$J\dot{\omega} = T$$





Wheeled Robot, 2

$$\left.\begin{aligned}\dot{x} &= v_x \cos\theta \\ \dot{y} &= v_x \sin\theta \\ \dot{\theta} &= \omega \\ \dot{v}_x &= F \\ \dot{\omega} &= T \\ y_1 &= x \\ y_2 &= y\end{aligned}\right\} \Rightarrow \begin{aligned}\ddot{y}_1 &= \dot{v}_x \cos\theta - v_x \sin\theta\,\omega = -v_x\omega\sin\theta + \cos\theta\, F \\ \ddot{y}_2 &= \dot{v}_x \sin\theta + v_x \cos\theta\,\omega = v_x\omega\cos\theta + \sin\theta\, F\end{aligned}$$

$$\alpha = \begin{bmatrix} -v_x\omega\sin\theta \\ v_x\omega\cos\theta \end{bmatrix}, \quad \rho = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}$$

$$\Rightarrow q = 1, \quad \alpha = \{1\} \quad \Rightarrow \text{add integrator in front of } u_1 = F$$

Wheeled Robot, 3

The extended system is:

$$\frac{d}{dt}\begin{bmatrix}\theta \\ x \\ y \\ v_x \\ \omega \\ F\end{bmatrix} = \begin{bmatrix}\omega \\ v_x\cos\theta \\ v_x\sin\theta \\ F \\ 0 \\ 0\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}u \\ T\end{bmatrix}$$

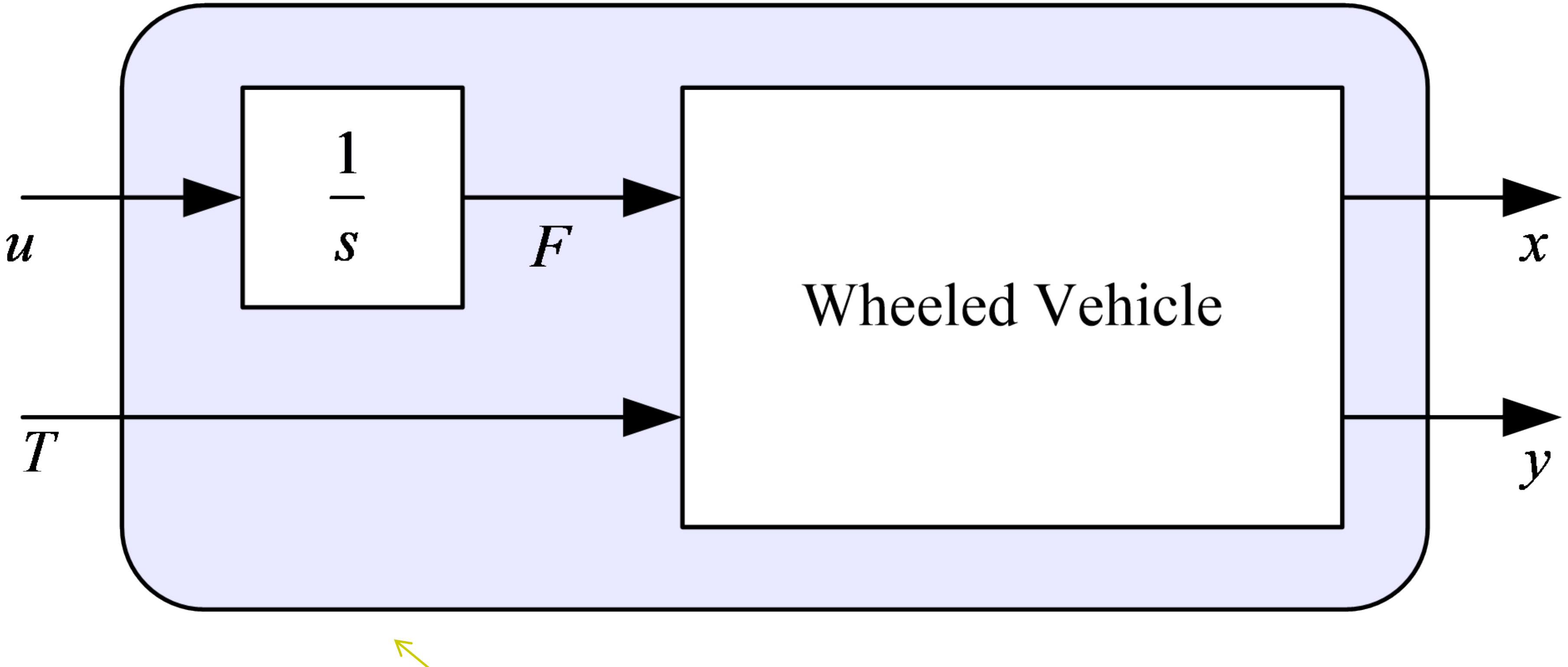
Wheeled Robot, 4

And, the normal form:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right]\left[\alpha + \rho u\right]$$

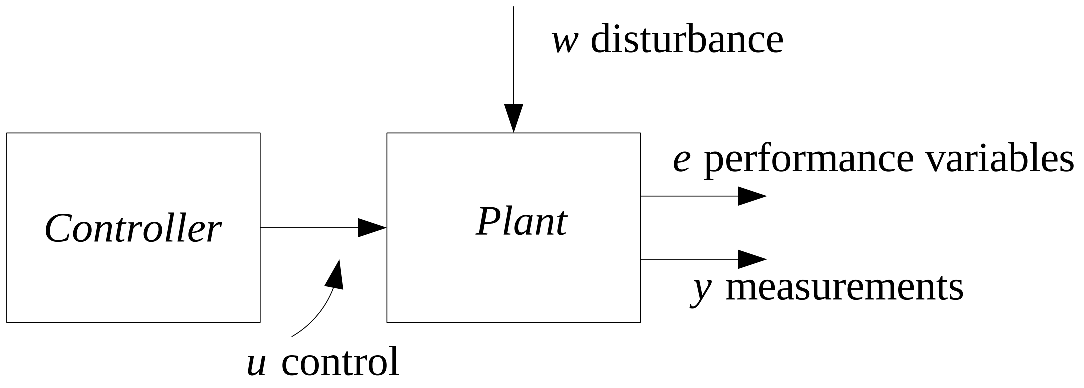
$$\alpha = \begin{bmatrix} -z\omega \sin\theta + \omega\left(-v_x \omega \cos\theta - z \sin\theta\right) \\ z\omega \cos\theta + \omega\left(-v_x \omega \sin\theta + z \cos\theta\right) \end{bmatrix}, \quad \rho = \begin{bmatrix} \cos\theta & -v_x \sin\theta \\ \sin\theta & v_x \cos\theta \end{bmatrix}$$

Wheeled Robot, 5



Feedback linearizable

Regulation of Feedback Linearizable Systems - 1



$\dot{x} = f(x,w) + G(x)u$ state

$\dot{w} = \varphi(w)$ disturbance

$e = h_e(x,w)$ error

$y = h_y(x,w)$ measurement

Assumptions:

- equilibrium point at the origin, i.e.
  $0 = f(0,0),\ 0 = h_z(0,0),\ 0 = h_y(0,0)$
- disturbance dynamics are neutrally stable

Goals:

1. stabilize the plant (internal stability)
2. regulate the performance variables: $\lim_{t \to \infty} e(t) = 0$

Regulation of Feedback Linearizable Systems - 2

Apply standard reduction to augmented system

$$(x, w) \rightarrow (\xi, z)$$

$$\dot{\xi} = F(\xi, w, z, u)$$

$$\dot{z} = A_0 z + E_0 \left[\alpha(x, w) + \rho(x, w) u \right]$$

$$e = C_0 z$$

choose: $u = \rho^{-1}(x, w)\left\{-\alpha(x, w) + v\right\}$

take $v = Kz(x, w)$ so that $\operatorname{Re}\lambda(A_0 + E_0 K) < 0$

$$\Rightarrow \lim_{t\to\infty} z(t) = 0 \Rightarrow \lim_{t\to\infty} e(t) = 0$$

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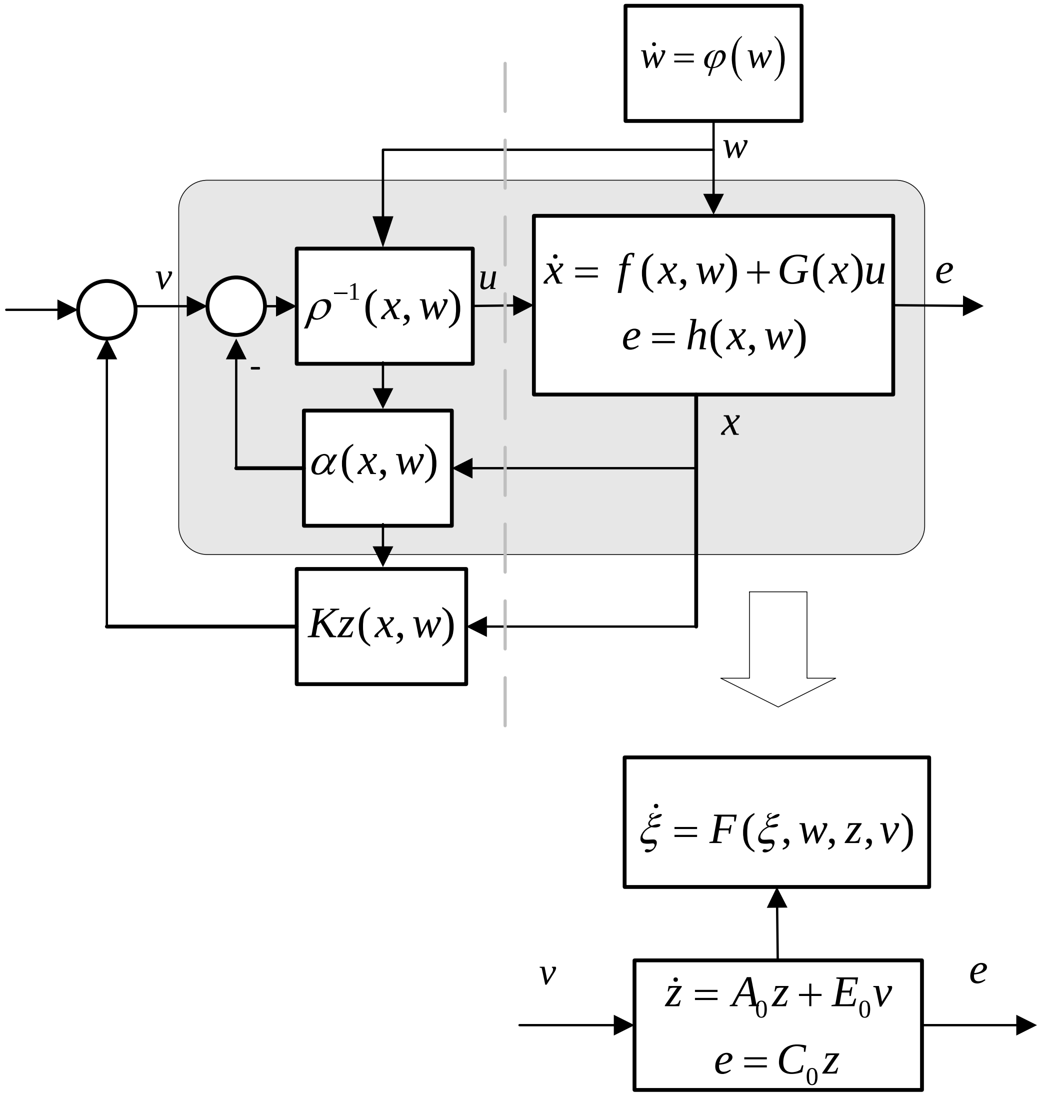
Regulation of Feedback Linearizable Systems - 3



Example: SISO Linear Systems

$$\dot{x} = Ax + Ew + bu$$

$$\dot{w} = Zw$$

$$z = cx + fw$$

$$\dot{z} = cAx + cEw + fZw + \cancel{cb}u = \begin{bmatrix} c & f \end{bmatrix} \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \cancel{cb}u$$

$$\ddot{z} = \begin{bmatrix} c & f \end{bmatrix} \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}^2 \begin{bmatrix} x \\ w \end{bmatrix} + \cancel{cAb}u$$

$$z^{(r)} = \begin{bmatrix} c & f \end{bmatrix} \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}^r \begin{bmatrix} x \\ w \end{bmatrix} + cA^{r-1}bu$$

First u appearance

$$\Rightarrow \rho = cA^{r-1}b, \alpha = \begin{bmatrix} c & f \end{bmatrix} \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}^r \begin{bmatrix} x \\ w \end{bmatrix}$$

Example, Continued

$$\alpha = \begin{bmatrix} cA^r & cE\left(A^{r-1} + A^{r-2}Z + \cdots + AZ^{r-2} + Z^{r-1}\right) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

expand matrix

$$\rho = cA^{r-1}b$$

$$u = \frac{1}{cA^{r-1}b}\left\{ -\begin{bmatrix} cA^r & cE\left(A^{r-1} + A^{r-2}Z + \cdots + AZ^{r-2} + Z^{r-1}\right) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + v \right\}$$

$$z_1 = \begin{bmatrix} c & f \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

$$z_2 = \begin{bmatrix} cA & cE + fZ \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

new z coordinates

$$\vdots$$

$$z_r = \begin{bmatrix} cA^{r-1} & cE\left(A^{r-2} + A^{r-3}Z + \cdots Z^{r-2}\right) + fZ^{r-1} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

Tracking: A Special Case

$$\dot{x} = f(x) + G(x)u, \quad y = h(x)$$

$$\dot{w} = Zw, \quad \bar{y} = Fw$$

$$e = y - \bar{y} = h(x) - \bar{y}$$

$$\dot{e} = \frac{\partial h}{\partial x}\dot{x} - \dot{\bar{y}} = L_f h(x) - \dot{\bar{y}} + L_g h(x)u$$

$$\text{if } L_g h(x) \neq 0 \text{ stop, else}$$

$$\ddot{e} = L_f^2 h(x) - \ddot{\bar{y}} + L_g L_f h(x)u$$

$$\text{if } L_g L_f^{k-1} h(x) \neq 0 \text{ stop, else continue until}$$

$$e^{(r)} = L_f^r h(x) - \bar{y}^{(r)} + L_g L_f^{r-1} h(x)u \text{ with } L_g L_f^{r-1} h(x) \neq 0$$

$$e = h(x) - \bar{y}$$
$$\dot{e} = L_f h(x) - \dot{\bar{y}}$$
$$\vdots$$
$$e^{(r-1)} = L_f^{r-1} h(x) - \bar{y}^{(r-1)}$$

note that: $\bar{y}^{(k)} = FZ^k w$

set $u = \dfrac{1}{L_g L_f^{r-1} h(x)}\left(-L_f^r h(x) + \bar{y}^{(r)} + v\right) \Rightarrow \boxed{e^{(r)} = v}$

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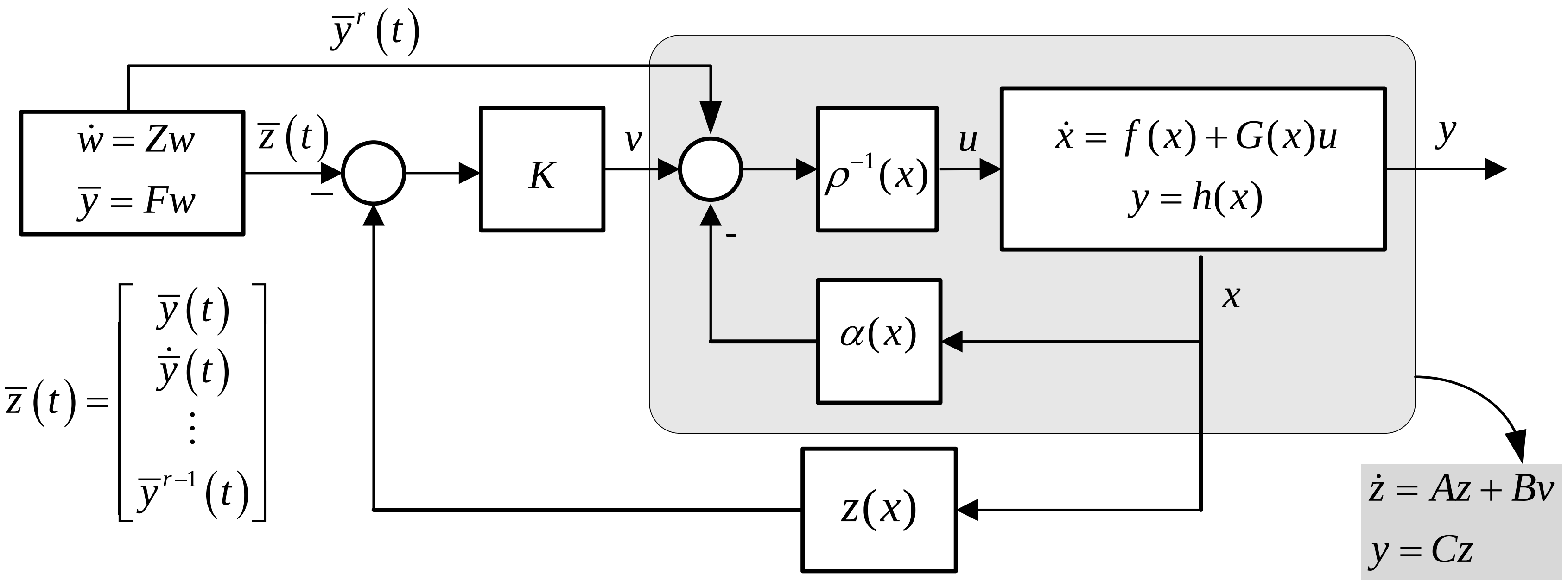
Tracking Regulator: General Structure



Example: Wheeled Robot Position Control

$$\dot{x} = v_x \cos\theta$$

$$\dot{y} = v_x \sin$$

$$\dot{\theta} = \omega$$

$$\dot{v}_x = F$$

$$\dot{\omega} = T$$

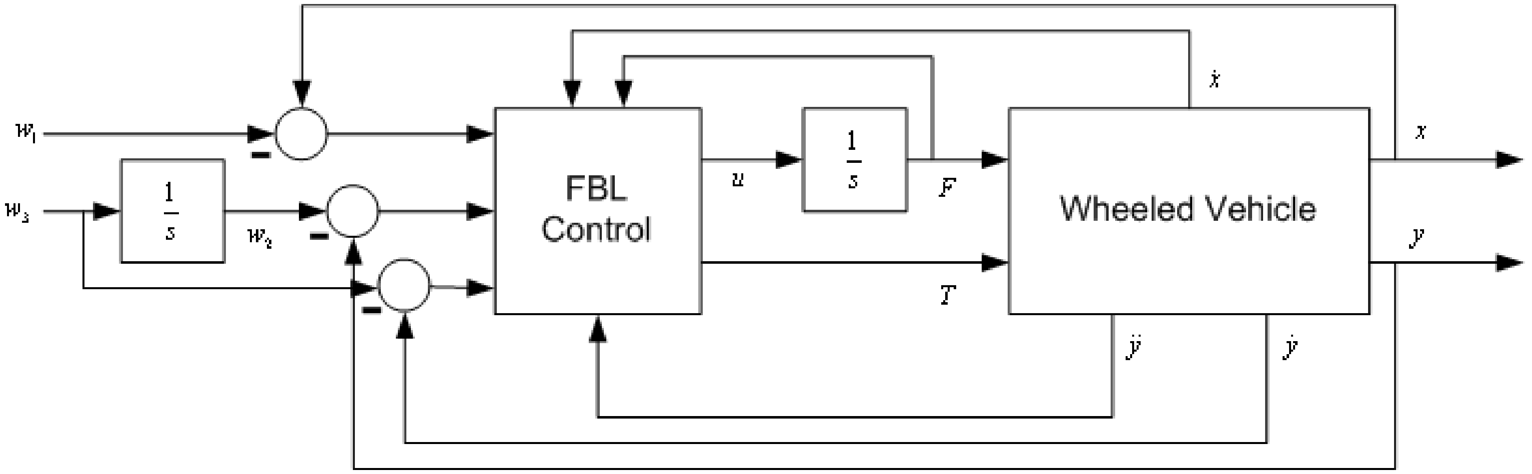
$$\dot{w}_1 = 0$$

$$\dot{w}_2 = w_3$$

$$\dot{w}_3 = 0$$

$$y_1 = x - w_1$$

$$y_2 = y - w_2$$



Note:

$$\bar{y}_1 = w_1, \dot{\bar{y}}_1 = 0$$

$$\bar{y}_2 = w_2, \dot{\bar{y}}_2 = w_3, \ddot{\bar{y}}_2 = 0$$

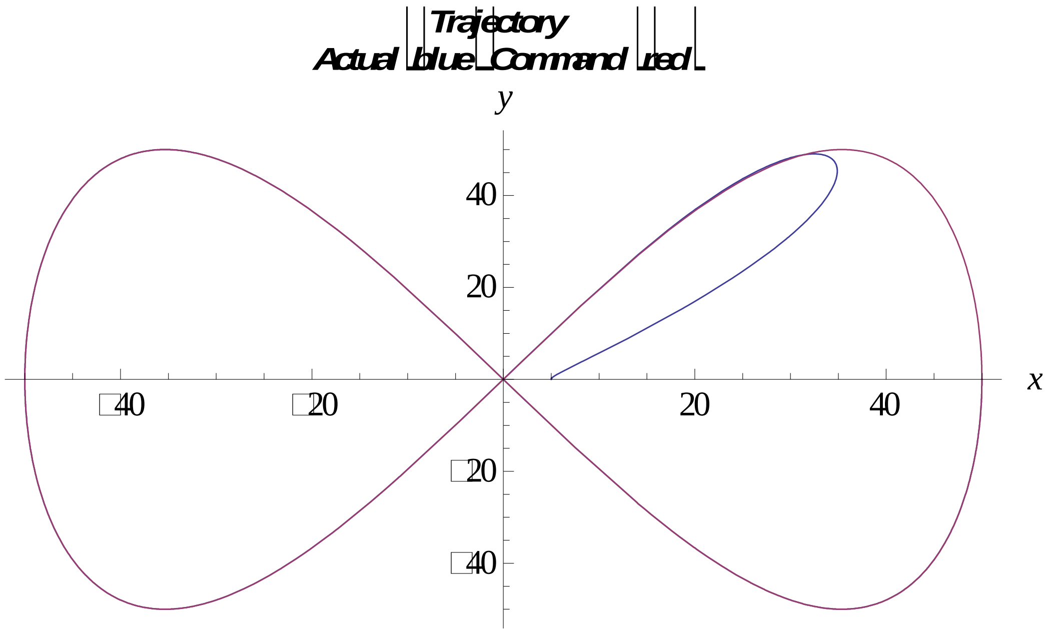
Recall:

$$r_1 = 3, r_2 = 3$$

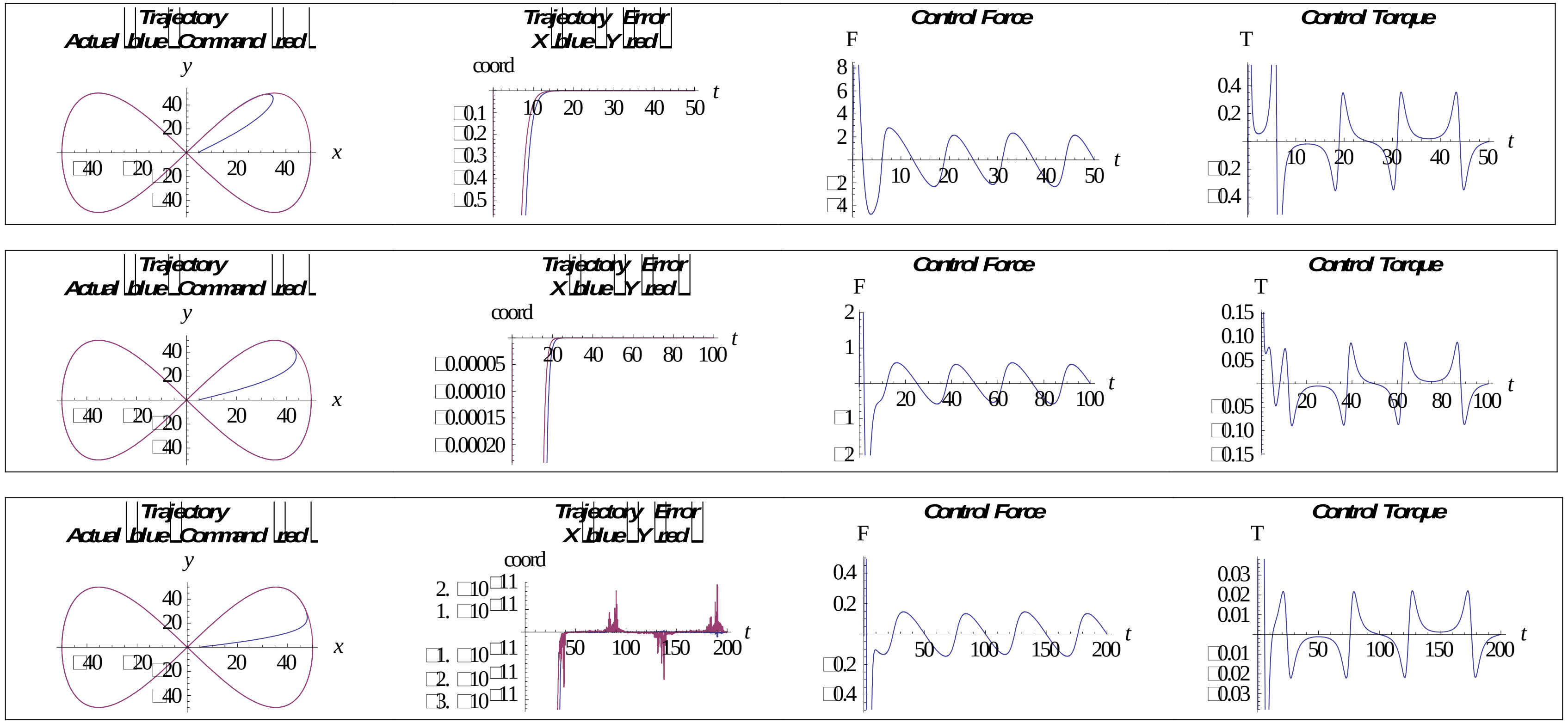
Fall 09 Project

$$x_c(t) = 50\cos\left(2\pi \frac{t}{\tau}\right)$$

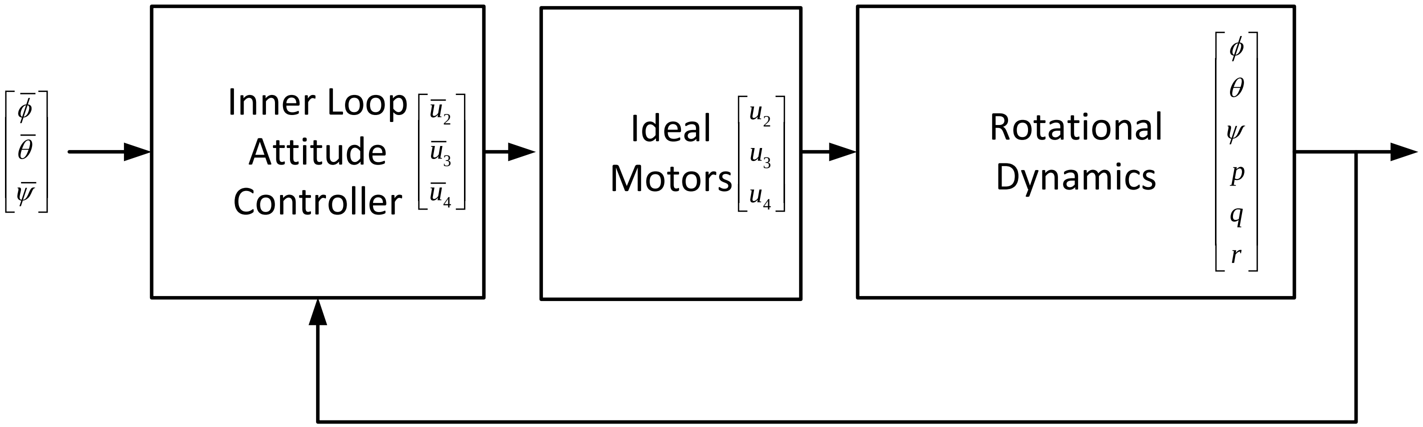
$$y_c(t) = 50\sin\left(4\pi \frac{t}{\tau}\right)$$



Fall 09 Project



Fall 2017 Project



$$\frac{d}{dt}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}=\begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}p\\ q\\ r\end{bmatrix}=\begin{bmatrix}\frac{I_y-I_z}{I_x}qr\\ \frac{I_z-I_x}{I_y}pr\\ \frac{I_x-I_y}{I_z}pq\end{bmatrix}+\begin{bmatrix}\frac{\ell}{I_x}u_2\\ \frac{\ell}{I_y}u_3\\ \frac{1}{I_z}u_4\end{bmatrix},\qquad y=\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}-\begin{bmatrix}\bar{\phi}\\ \bar{\theta}\\ \bar{\psi}\end{bmatrix}=\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}-\begin{bmatrix}w_5\\ w_3\\ w_1\end{bmatrix}$$

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$$\dot{y} = \frac{d}{dt}\begin{bmatrix}\phi \\ \theta \\ \psi\end{bmatrix} - \frac{d}{dt}\begin{bmatrix}w_5 \\ w_3 \\ w_1\end{bmatrix} = \begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{bmatrix}\begin{bmatrix}p \\ q \\ r\end{bmatrix} - \begin{bmatrix}0 \\ w_4 \\ w_2\end{bmatrix}$$

$$\ddot{y} = \frac{d}{dt}\left(\begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{bmatrix}\begin{bmatrix}p \\ q \\ r\end{bmatrix}\right) - \begin{bmatrix}0 \\ -w_3 \\ 0\end{bmatrix}$$

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