Discontinuous Systems: Intro to Switching Control

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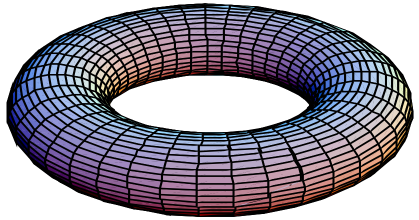

Part II Outline

- Intro to Discontinuous Dynamics
  - Examples
  - Simulation Tools
  - Solution concepts
- Variable Structure Control Basics
  - Sliding domain, equivalent control
  - Lyapunov analysis of discontinuous systems
  - Special Cases: linear dynamics, normal from
- Hybrid Systems
  - Mixed logic-dynamic systems
  - Modeling using Simulink with State Flow
  - Logic to mixed-integer formulas
  - Optimization

Outline

- Brocket’s Necessary Condition
  - Some systems cannot be stabilized by smooth state feedback
  - Extensions to BNC
- Solutions to Discontinuous Differential Equations
  - Various notions of ‘solution ‘may be appropriate

What is a Discontinuous System

Consider a system

$$\dot{x} = f(x), \quad x \in R^n$$

such a system is considered to be a continuous system if the function $f(x)$ has continous first derivatives in x, otherwise it is discontinuous.

A control system is more complicated.

$$\dot{x} = f(x,u), \quad x \in R^n, u \in R^m, f(0,0) = 0$$

A control system is considered to be a continuous if the function $f(x,u(x))$ has continous first derivatives in x, otherwise it is discontinuous.

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Brockett's Necessary Condition

Necessary Condition for Asymptotic Stability

$$\dot{x} = f(x,u), \quad x \in R^n, u \in R^m, f(0,0) = 0$$

Theorem: (Brockett) Suppose f is smooth and the origin is stabilized by a smooth state feedback control $u(x)$, $u(0) = 0$. Then the mapping $F : R^n \to R^n$, $F(x) = f(x, u(x))$ maps neighborhoods of the origin into neighborhoods of the origin, i.e.

$$\forall \delta > 0 \quad \exists \varepsilon > 0 \text{ such that } B_\varepsilon \subset F(B_\delta)$$

alternatively, $f(B_\delta \times R^m)$ is a neighborhood of $0 \in R^n$.

Example 1

$$\dot{x}_1 = u(x_1, x_2),$$

$u(x_1, x_2)$, smooth and $u(0,0) = 0$

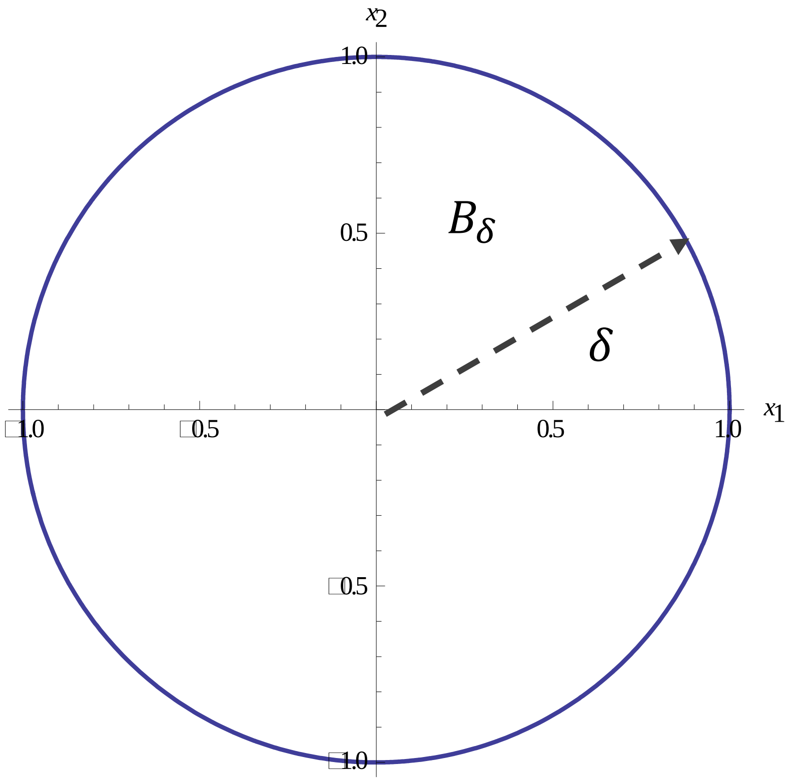
$$\dot{x}_2 = x_1 u(x_1, x_2)$$

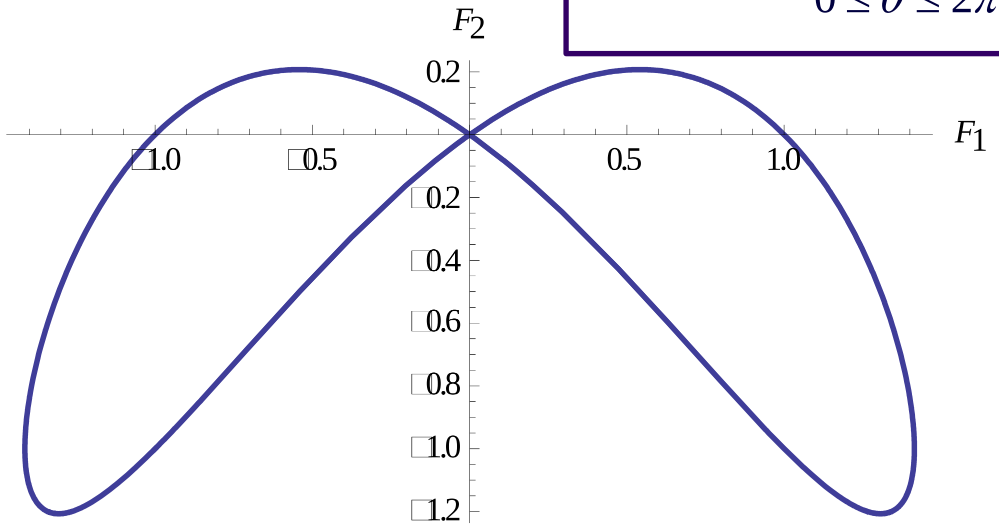
$$F = \begin{bmatrix} u(x_1, x_2) \\ x_1 u(x_1, x_2) \end{bmatrix}$$

The only point of image on F2 axis is the origin.

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} u(\delta \sin\theta, \delta\cos\theta) \\ \delta \sin\theta u(\delta\sin\theta, \delta\cos\theta) \end{bmatrix}$$

$$0 \le \theta \le 2\pi$$





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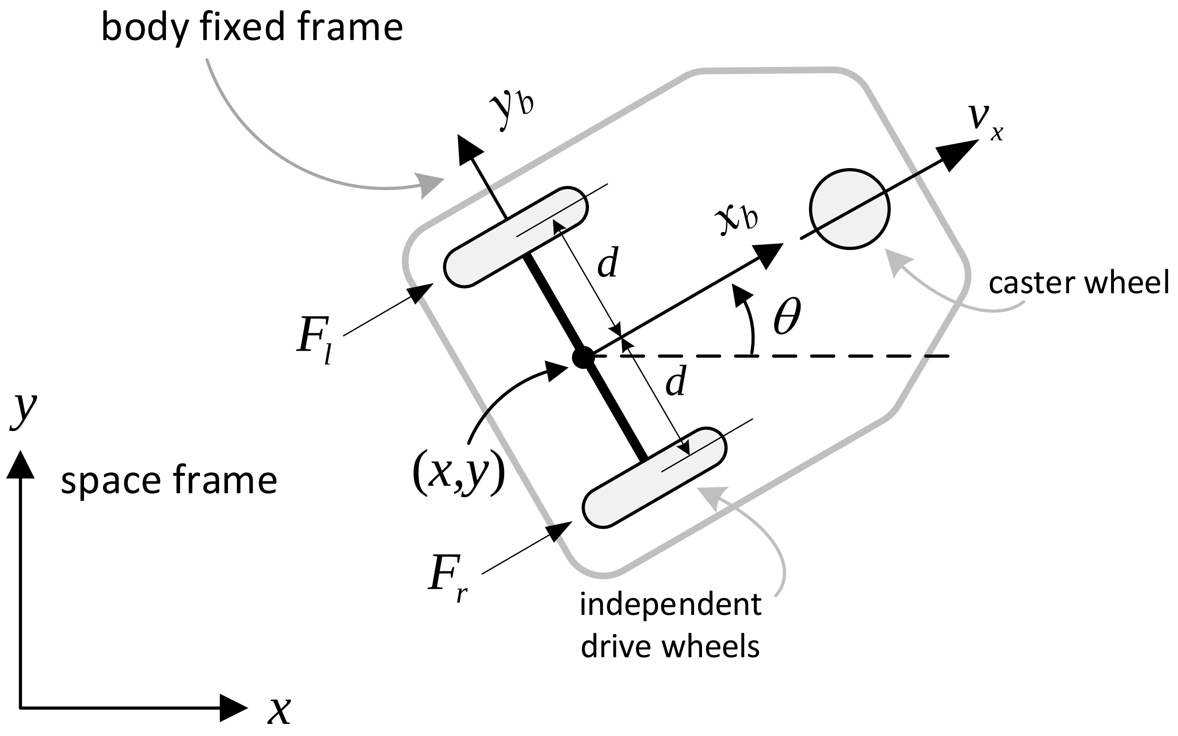
Example 2

$$\begin{aligned} \dot{x} &= v_x \cos\theta \\ \dot{y} &= v_x \sin\theta \\ \dot{\theta} &= \omega \end{aligned} \quad \Leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 \, u_1 \\ \sin x_3 \, u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$



Notice that with $x_3 = 0$, all points on the F_2 axis other than 0 are not in the image of the mapping.

Notions of Solution for Discontinuous Dynamics

Solutions of ODEs

- Classical Solutions

$$\dot{x}(t) = f\left(x(t), t\right), \quad x(0) = x_0$$

- Caratheodory Solutions

$$x(t) = x_0 + \int_0^t f\left(x(s), s\right) ds$$

  - Satisfies the ode almost everywhere on [0,*t*], i.e $\dot{x}(t) \neq f\left(x(t), t\right)$ at isolate points of time.

- Filippov Solutions (differential inclusion a set)

$$\dot{x}(t) \in \mathfrak{F}\left(x(t), t\right), \quad x(0) = x_0$$

Classical Solutions

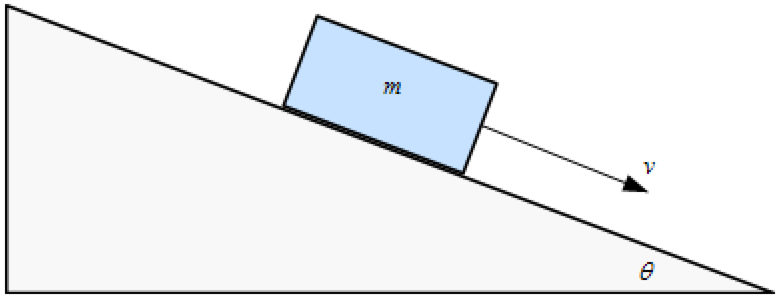
$$\dot{x}(t) = f\left(x(t), t\right)$$

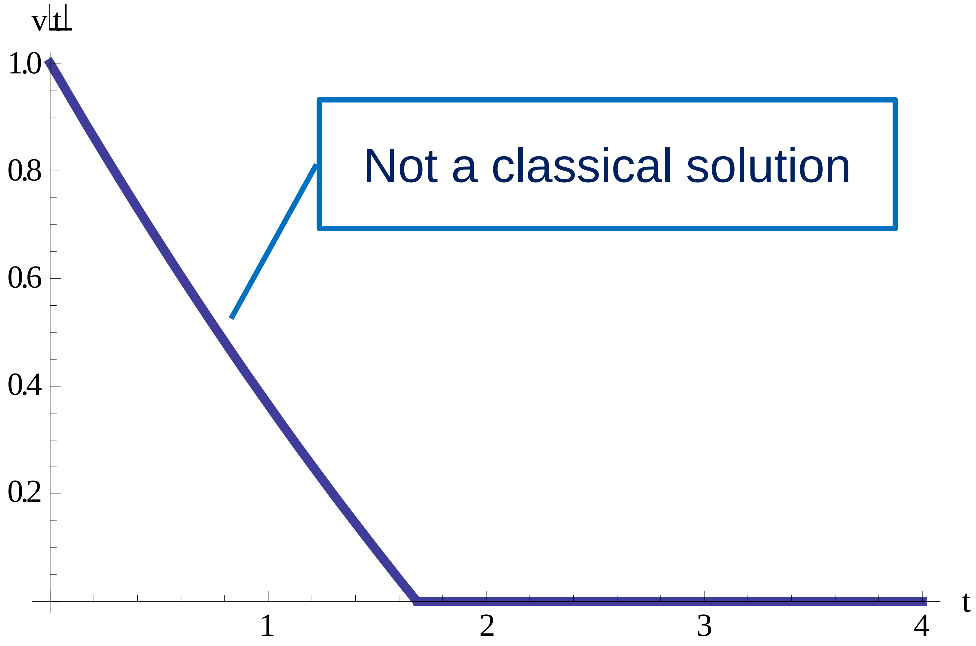
classical solution: $x(t)$ is continuously differentiable.

Example: brick on ramp

with stiction.

$m\dot{v} = -\kappa \operatorname{sgn} v - cv + mg \sin\theta$





Caratheodory Solutions

$\dot{x}(t) = f(x(t))$ is satisfied at almost all points on every interval $t \in [a,b]$, $a < b$

Stopping solutions for the brick on ramp problems are not Caratheodory solutions. For these solutions the brick is stopped on a finite interval, i.e, $v(t) = 0$ on $t \in [a,b]$

$\Rightarrow \dot{v}(t) = 0$ on $t \in [a,b]$

$\Rightarrow -\kappa \operatorname{sgn} 0 + mg \sin\theta = 0$

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Brick Example – try something else

$$m\dot{v} = -\kappa \,\mathrm{sgn}\, v - cv + mg\sin\theta \Rightarrow$$

$$\dot{v} = -\frac{\kappa}{m}\mathrm{sgn}\, v - \frac{c}{m}v + g\sin\theta$$

$$\dot{v} = -\frac{\kappa}{m} - \frac{c}{m}v + g\sin\theta \qquad v > 0$$

$$\dot{v} \in \left[-\frac{\kappa}{m} + g\sin\theta, \frac{\kappa}{m} + g\sin\theta\right] \qquad v = 0$$

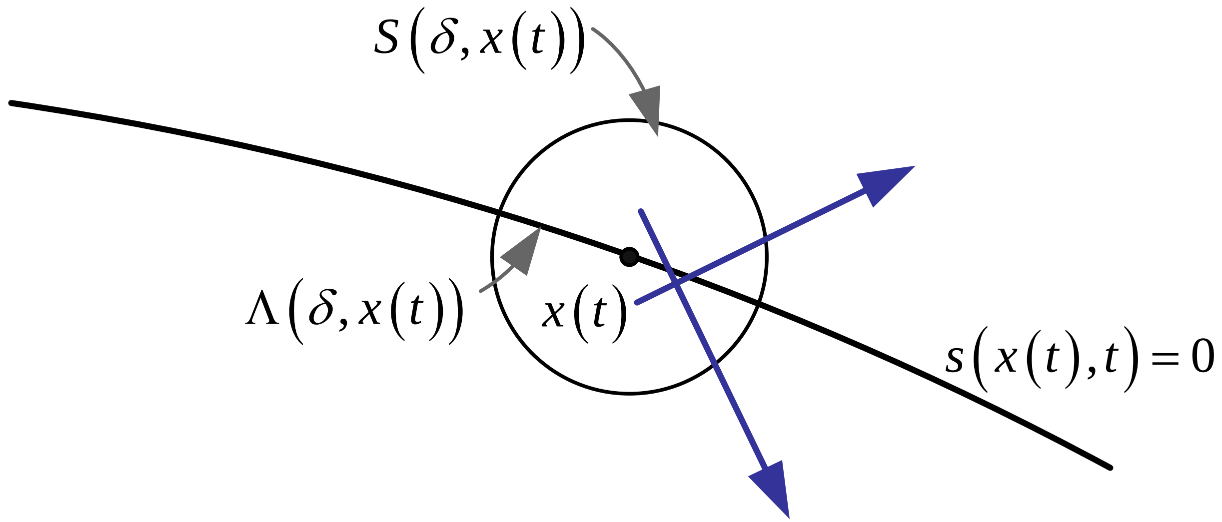
$$\dot{v} = \frac{\kappa}{m} - \frac{c}{m}v + g\sin\theta \qquad v < 0$$

Filippov Solutions

$$\frac{dx(t)}{dt} \in \mathfrak{F}(x(t),t) := \bigcap_{\delta>0} \operatorname{conv} f\left(S(\delta, x(t)) - \Lambda(\delta, x(t)), t\right)$$

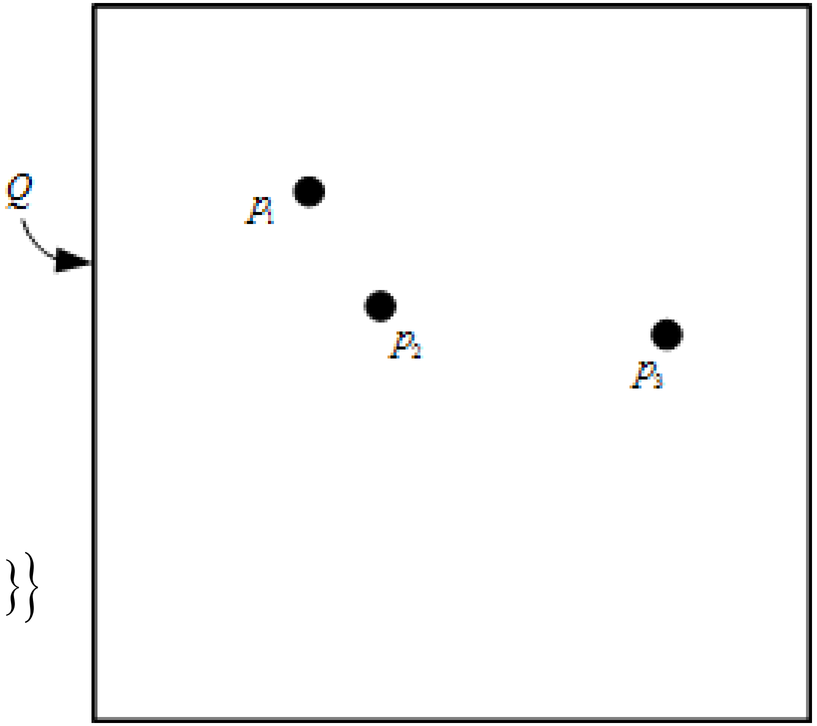
$$S(\delta, x) := \left\{ y \in R^n \,\middle|\, \|y - x\| < \delta \right\}$$

$\Lambda(\delta, x)$: subset of measure zero on which f is not defined



Example: nearest neighbor

- 3 agents moving in square *Q*
- Rule: move diametrically away from nearest neighbor



Nearest neighbor to p_i

$$\mathcal{N}_i = \arg\min \left\{ \|p_i - q\| \;\middle|\; q \in \partial Q \cup \{p_1, p_2, p_3\} \setminus \{p_i\} \right\}$$

Action

$$\dot{p}_i = \frac{p_i - \mathcal{N}_i}{\|p_i - \mathcal{N}_i\|}$$

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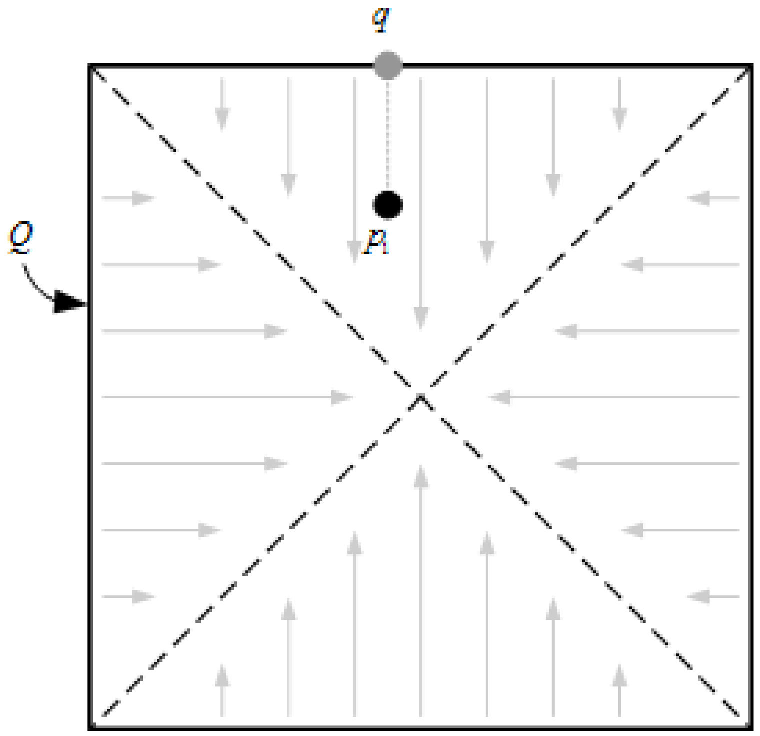
Example: nearest neighbor, cont'd

- Consider 1 agent - in which case the only obstacles are the walls.
- The nearest neighbor is easily identified on the nearest wall.
- The vector field is well defined everywhere except on the diagonals where it is not defined because there are multiple nearest neighbors.

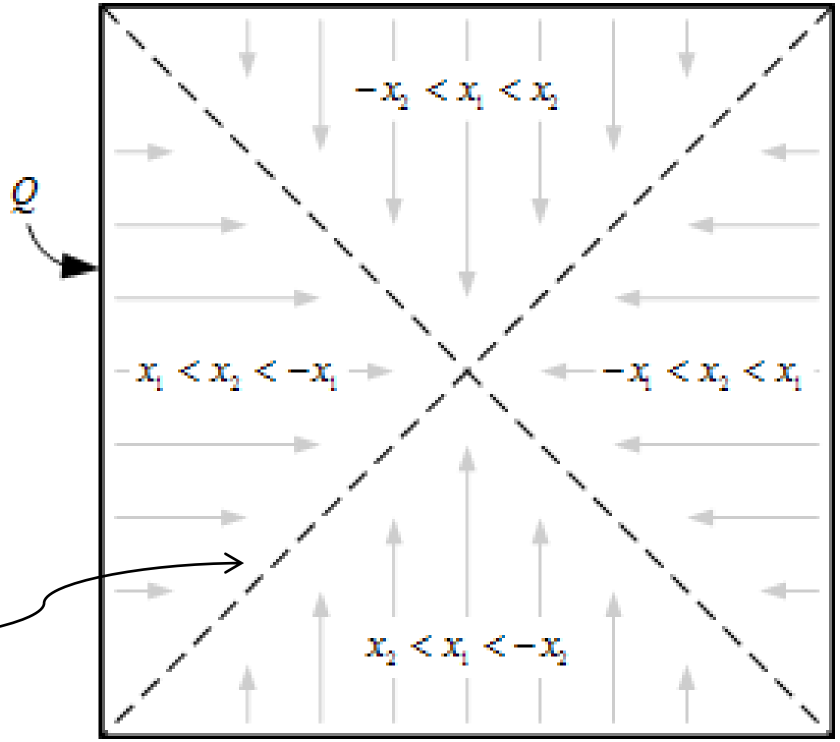


$$\dot{p}_1 = \frac{p_1 - q}{\|p_1 - q\|}$$

Example: nearest neighbor, cont'd

$$\dot{x} = f(x_1, x_2) = \begin{cases} (0,-1) & -x_2 < x_1 < x_2 \\ (-1,0) & -x_1 < x_2 < x_1 \\ (0,1) & x_2 < x_1 < -x_2 \\ (1,0) & x_1 < x_2 < -x_1 \end{cases}$$

$$f \in \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 0 \le \alpha \le 1$$



Extension of Brockett's Condition

Assumption on $f(x,u)$:

$$A \subseteq R^m \text{ convex} \Rightarrow f(x,A) \subseteq R^n \text{ convex}$$

$$\Updownarrow$$

$$\forall A \subseteq R^m \ \text{conv}\, f(x,A) \subseteq f(x, \text{conv}\, A)$$

Definition: Admissible feedback controls $u(x)$ are piecewise continuous and solutions are defined in the sense of Filippov

$$\dot{x} \in \mathcal{F}\left[f\left(x, u(x)\right)\right] \text{ and } 0 \in \mathcal{F}\left[f\left(0, u(0)\right)\right]$$

Theorem (Ryan): For $f(x,u)$ continuous and satisfying assumption, asymptotic stabilization by discontinuous feedback $\Rightarrow$ each neighborhood $\mathcal{B}$ of $0 \in R^n$, $f\left(\mathcal{B} \times R^m\right)$ is a neighborhood of $0 \in R^n$.