Sliding Modes

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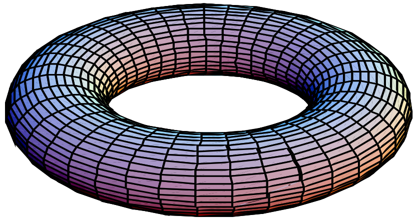

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Outline

- A preliminary example
- Basics of switching control systems
- Supplemental Material on Euclidean Geometry

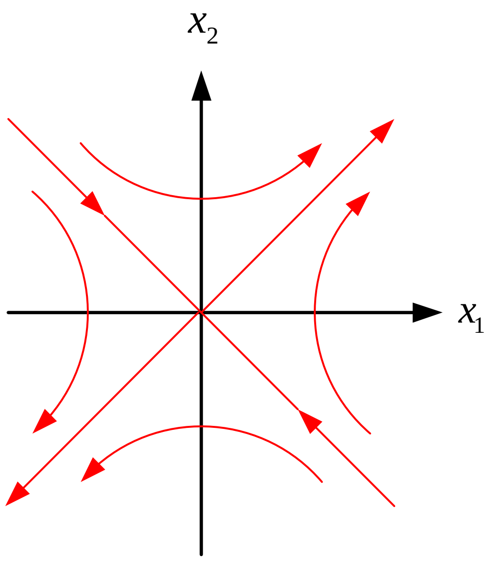
PRELIMINARY EXAMPLE

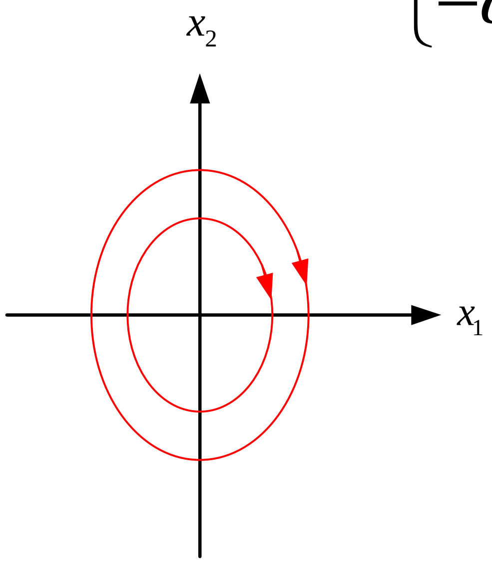
Example

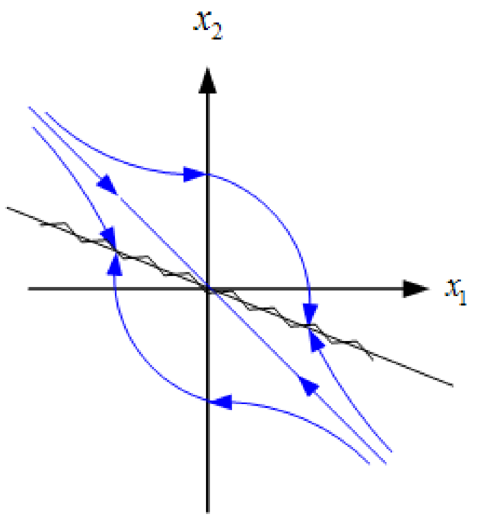
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = ax_1 - bu, \quad a, b > 0$$

$$u = \begin{cases} \alpha x_1 & x_1 s > 0 \\ -\alpha x_1 & x_1 s < 0 \end{cases}, \quad s = cx_1 + x_2 \quad \alpha, c > 0$$







sliding mode: $s = cx_1 + x_2 \equiv 0 \Rightarrow \dot{x}_1 + cx_1 = 0$

condition for existence: $s\dot{s} < 0$

SWITCHING CONTROL BASICS

Setup

System:

$$\dot{x} = f(x, u)$$

$$x \in R^n, u \in R^m, \ f \text{ smooth}$$

Control:

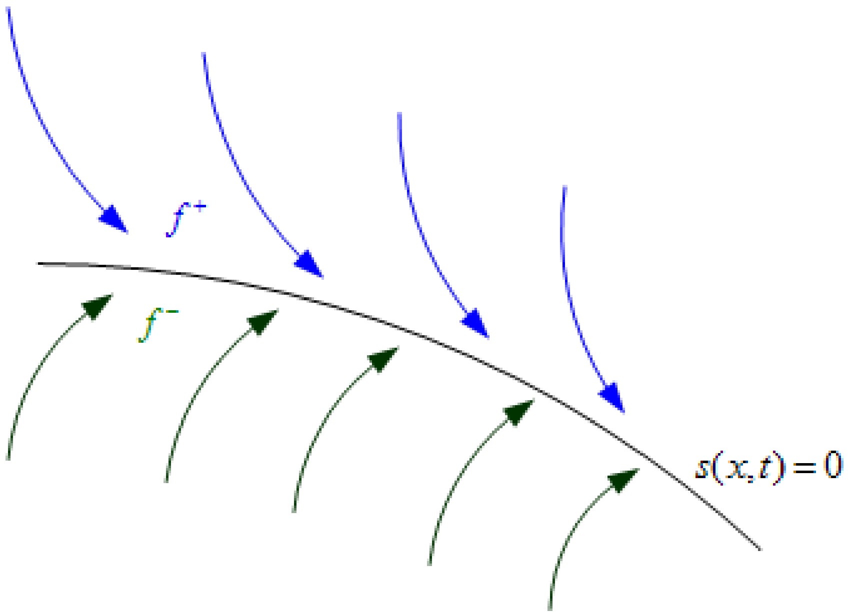
$$u_i \text{ discontinuous across switching surface } s_i(x) = 0$$

$$u_i(x) = \begin{cases} u_i^+(x), & s_i(x) > 0 \\ u_i^-(x), & s_i(x) < 0 \end{cases}, \quad i = 1, \ldots, m$$

$$u_i^+(x), \ u_i^-(x), \text{ smooth}$$

Switching Systems & Sliding Motion

$$\dot{x} = f(x,t) = \begin{cases} f^{+}(x,t) & s(x,t) > 0 \\ f^{-}(x,t) & s(x,t) < 0 \end{cases}$$



If there is a motion in the switching surface, it is called a sliding motion.

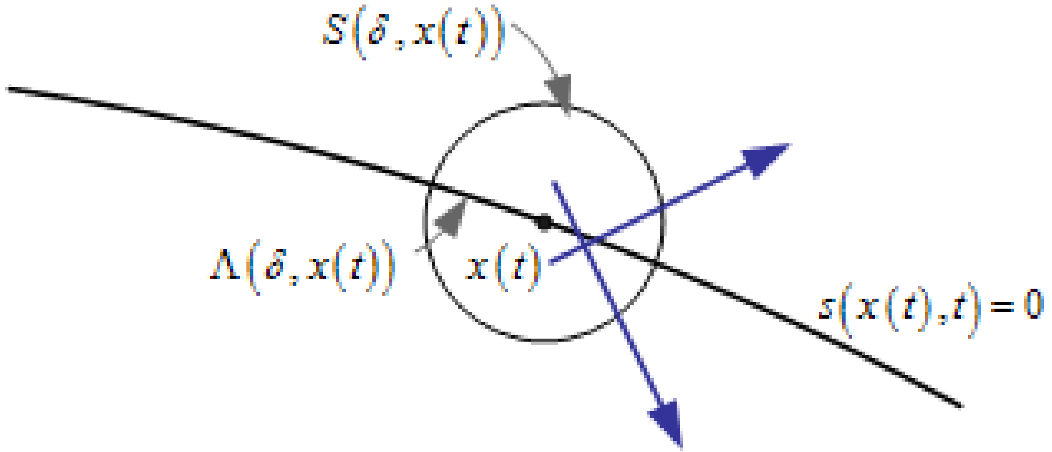
How is it defined?

Defining the Sliding Motion via The Filippov Solution

$$\frac{dx(t)}{dt} \in \mathfrak{F}(x(t),t) := \bigcap_{\delta>0} \operatorname{conv} f\left(S(\delta, x(t)) - \Lambda(\delta, x(t)), t\right)$$

$$S(\delta, x) := \left\{ y \in R^n \,\middle|\, \|y - x\| < \delta \right\}$$

$\Lambda(\delta, x)$: subset of measure zero on which f is not defined



Lyapunov Stability for Discontinuous Vector Fields

Using smooth Lyapunov functions with discontinuous vector fields.

$$V(x) \in C^1$$

$$\dot{V}(x(t)) \in \left\{ \frac{\partial V}{\partial x}\xi \,\middle|\, \xi \in \mathscr{F}(x(t),t) \right\}$$

Result 1

$$\dot{V} \le -\rho < 0 \left(\dot{V} \ge \rho > 0\right) \text{ on } P - \Lambda \Rightarrow$$

$$\dot{V} \le -\bar{\rho} < 0 \left(\dot{V} \ge \bar{\rho} > 0\right) \text{ on } P$$

Open region of state space

Surface of discontinuity

Result 2

$$\dot{V} \le -\rho \|s(x)\|,\ \rho > 0 \text{ on } P - \Lambda \Rightarrow \dot{V} \le -\rho \|s(x)\|,\ \rho > 0 \text{ on } P$$

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Sliding Domain

$D_s \subset M_s = \left\{ x \in R^n \,\middle|\, s(x) = 0 \right\}$ is a sliding domain if:

1. trajectories beginning in a δ-vicinity of D_s remain in an ε-vicinity until reaching ∂D_s

2. D_s does not contain entire trajectories of the 2^m associated continuous systems.

See notes in Chapter 10

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Proposition

Lyapunov verification of a sliding domain.

$$V(x) \in C^1, \quad V(x) := \begin{cases} = 0 & if \quad s(x) = 0 \\ > 0 & otherwise \end{cases}$$

$D \supset D_s$ is an open, connected subset of R^n

$$\dot{V} \le -\rho \left\| s(x) \right\| < 0 \text{ on } D - M_s$$

$$\Downarrow$$

D_s is a sliding domain

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Proposition

Finite time reaching of sliding domain.

$$V(x) = \sigma \left\| s(x) \right\|^2, \sigma > 0 \text{ on a } \delta\text{-vicinity of } D_s$$

$$D \supset D_s \text{ is an open, connected subset of } R^n$$

$$\dot{V} \leq -\rho \left\| s(x) \right\| < 0 \text{ on } D - M_s$$

$$\Downarrow$$

trajectories beginning in a δ-vicinity of D_s *reach* D_s in finite time

SUPPLEMENTARY MATERIAL: EUCLIDEAN GEOMETRY

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Geometry

- A subset $\mathscr{S}$ of the linear space (over field $\mathbb{F}$) $\mathscr{X}$ is a linear subspace of $\mathscr{X}$ if:

  $\forall x_1, x_2 \in \mathscr{S}$ and

  $\forall c_1, c_2 \in \mathbb{F}, c_1 x_1 + c_2 x_x \in \mathscr{S}$

For our purposes think Euclidean space R^n

- If $x_i \in \mathscr{X}$ $(i = 1, \ldots, k)$, then $\text{span}\{x_1, \ldots, x_k\}$ is a subspace of $\mathscr{X}$.
- $\mathscr{R}, \mathscr{S} \subset \mathscr{X}$ then

  $\mathscr{R} + \mathscr{S} = \{r + s \mid r \in \mathscr{R}, s \in \mathscr{S}\}$

  $\mathscr{R} \cap \mathscr{S} = \{x \mid x \in \mathscr{R} \;\&\; x \in \mathscr{S}\}$

- Two subspaces $\mathscr{R}, \mathscr{S}$ are independent if $\mathscr{R} \cap \mathscr{S} = 0$

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Geometry 2

- If $\mathscr{R}_i, i = 1, \ldots, k$ are independent subspaces, then the sum

$$\mathscr{R} = \mathscr{R}_1 + \cdots + \mathscr{R}_k$$

is called an indirect sum and may be written

$$\mathscr{R} = \mathscr{R}_1 \oplus \cdots \oplus \mathscr{R}_k$$

The symbol $\oplus$ presupposes independence.

- Let $\mathscr{X} = \mathscr{R} \oplus \mathscr{S}$. For each $x \in \mathscr{X}$, there are unique $r \in \mathscr{R}, s \in \mathscr{S}$ so that $x = r + s$. This implies a unique function $x \mapsto r$ called the projection on $\mathscr{R}$ along $\mathscr{S}$.

Geometry 3

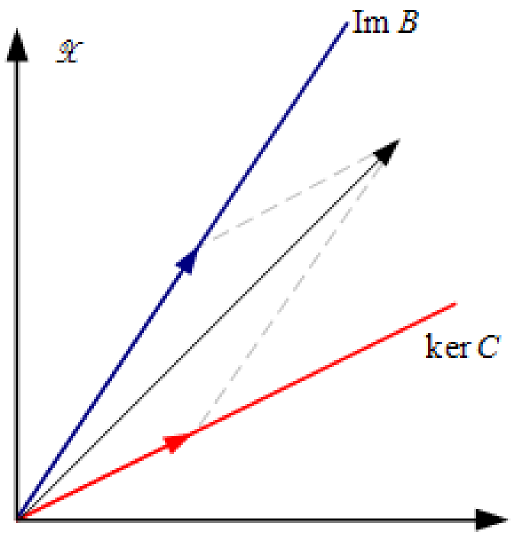
- The projection is a linear map $Q:\mathscr{X}\to\mathscr{X}$, such that $\operatorname{Im} Q=\mathscr{R}$ and $\ker Q=\mathscr{S}$, and $\mathscr{X}=Q\mathscr{X}\oplus(I-Q)\mathscr{X}=\mathscr{R}\oplus\mathscr{S}$
- Note that $(I-Q)$ is the projection on $\mathscr{S}$ along $\mathscr{R}$. Thus, $Q(I-Q)=0\Leftrightarrow Q^2=Q$
- Conversly, for any map $Q:\mathscr{X}\to\mathscr{X}$ such that $Q^2=Q$ $\mathscr{X}=\operatorname{Im} Q\oplus\ker Q$

i.e., Q is the projection on $\operatorname{Im} Q$ along $\ker Q$.

Geometry 4

Recall

$$\dot{x} = \left[I - B(CB)^{-1} C \right] Ax$$



$$Q = B(CB)^{-1} C$$

$$Q^2 = B(CB)^{-1} CB(CB)^{-1} C = B(CB)^{-1} C$$

$$\operatorname{Im} Q = \operatorname{Im} B \quad \ker Q = \ker C \quad X = \operatorname{Im} B \oplus \ker C$$

$B(CB)^{-1} C$ is the projection on $\operatorname{Im} B$ along $\ker C$

$\left(I - B(CB)^{-1} C\right)$ is the projection on $\ker C$ along $\operatorname{Im} B$