VSC Design via Sliding Modes

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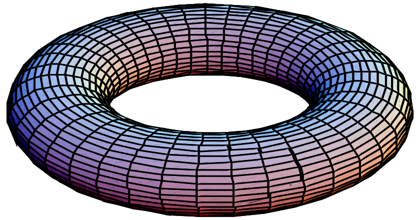

Outline

- VS systems, sliding modes, reaching
- Example: undersea vehicle
- Design based on normal form

VSC Design via Sliding Modes: Setup

System:

$$\dot{x} = f(x,u)$$

$$x \in R^n, u \in R^m, \ f \text{ smooth}$$

Control:

$$u_i \text{ discontinuous across switching surface } s_i(x) = 0$$

$$u_i(x) = \begin{cases} u_i^+(x), & s_i(x) > 0 \\ u_i^-(x), & s_i(x) < 0 \end{cases}, \quad i = 1, \ldots, m$$

$$u_i^+(x),\ u_i^-(x), \text{ smooth}$$

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VSC Design via Sliding Modes: Strategy

1. Choose switching surfaces, $s_i(x)$, so that sliding mode has desired dynamics.
2. Choose control functions, $u_i^{\pm}(x)$, so that sliding mode is reached in finite time.

Approach:

1. How does choice of $s_i(x)$ affect sliding behavior?

   $\Rightarrow$ equivalent control method

2. How can we choose $u_i^{\pm}(x)$ to insure finite time reaching?

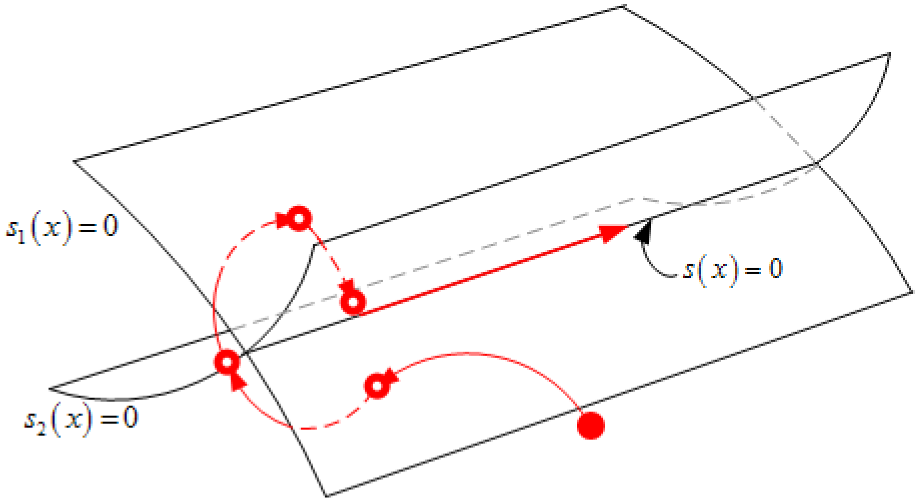
   $\Rightarrow$ Lyapunov method

Equivalent Control ~ 1

$$\dot{x} = f(t,x,u), \quad u_i = \begin{cases} u_i^+(t,x) & s_i(x) > 0 \\ u_i^-(t,x) & s_i(x) < 0 \end{cases}$$

Suppose $M_s = \{x \in R^n \,|\, s(x) = 0\}$ contains a sliding domain.

We can obtain the dynamics in M_s by identifying the equivalent control.



Equivalent Control ~ 2

If there exists a control $u(t,x)$ such that

$\dot{x} = f(t,x,u)$ and $s(x) = 0,$ then u is referred

to as the equivalent control and denoted u_{eq}

$$\dot{s}(x) = \frac{\partial s}{\partial x} f(t,x,u_{eq}) \equiv 0$$

if a solution exists for u_{eq}, then on $s = 0,\ \dot{s} \equiv 0$

and

$$\dot{x} = f(t,x,u_{eq}(t,x)) \quad \Leftarrow \text{dynamics in sliding}$$

Equivalent Control – Special Cases

Systems 'linear in control'

$$\dot{x} = f(x) + G(x)u$$

$$\dot{s}(x) = S(x)f(x) + S(x)G(x)u_{eq} \equiv 0$$

$$\det SG \neq 0 \Rightarrow u_{eq} = -\left[S(x)G(x)\right]^{-1} S(x)f(x)$$

$$\dot{x} = \left[I - G(x)\left[S(x)G(x)\right]^{-1} S(x)\right] f(x)$$

Linear Systems

$$\dot{x} = Ax + Bu, \quad s(x) = Cx$$

$$u_{eq} = (CB)^{-1} CAx \quad (\text{assuming } \det CB \neq 0)$$

$$\dot{x} = \left[I - B(CB)^{-1} C\right] Ax$$

Reaching

Choose

$$V(x) = s^T(x)s(x) = \|s(x)\|^2 \Rightarrow$$

$$\dot{V}(x) = 2s^T(x)\frac{\partial s}{\partial x} f(x,u)$$

Suppose $2s^T(x)(\partial s/\partial x) f(x,u) < -\rho\|s(x)\|$ on $D - M_s$ where

D is an open set containing $D_s \subset M_s \Rightarrow$

D_s is a sliding domain and it is reached in finite time

from any initial point in D.

Linear Example ~ Reaching

$$\dot{x} = f(x,u) = Ax + bu,\ s(x) = cx$$

$$V = \|s(x)\|^2 \Rightarrow \dot{V} = 2s(x)cAx + 2s(x)cbu$$

We can only affect the second term. Choose,

$$u = -k(x)\operatorname{sgn}(s(x)cb), \quad k(x) > 0 \forall x$$

$$\dot{V} = 2s(x)\left[cAx - cbk(x)\operatorname{sgn}(s(x)cb)\right] \le -\rho|s(x)|$$

provided $cbk(x) - |cAx| \ge \rho, \quad \forall x$

$$k(x) = (\rho + |cAx|)/(cb)$$

This insures that $s(x)$ is a global sliding domain.

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Linear SISO Example ~ 1

$$s(x) = cx, \quad s \equiv 0 \Rightarrow \dot{s} \equiv 0$$

$$c\dot{x} = cAx + cbu_{eq} := 0 \Rightarrow \boxed{u_{eq} = -(cb)^{-1} cAx}$$

$$\dot{x} = \left[I - b(cb)^{-1} c \right] Ax$$

Sliding dynamics

Define a matrix V whose columns span $\ker c$, i.e.,

$$V = [v_1 \quad \cdots \quad v_{n-1}], cv_i = 0$$

Notice that $b \notin \ker c$ and $X = \operatorname{Im} b \oplus \ker c$. Define a state transformation

$$x \mapsto (w, z) \quad x = Vw + bz, \quad w \in R^{n-1}, z \in R$$

$$V\dot{w} + b\dot{z} = \left[I - b(cb)^{-1} c \right] A(Vw + bz)$$

Linear SISO Example ~ 2

$$\begin{bmatrix} V & b \end{bmatrix} \begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \left[I - b(cb)^{-1} c \right] A \begin{bmatrix} V & b \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}$$

$$\begin{bmatrix} V & b \end{bmatrix}^{-1} = \begin{bmatrix} U \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} UV & Ub \\ dV & db \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow d = (cb)^{-1} c$$

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} U \\ d \end{bmatrix} \left[I - b(cb)^{-1} c \right] A \begin{bmatrix} V & b \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix} A \begin{bmatrix} V & b \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}$$

note:

$$d \left[I - b(cb)^{-1} c \right] = 0$$

$$\boxed{\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} UAV & UAb \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}}$$

sliding: $z \to 0$

$$\dot{w} = \left[UAV \right] w$$

Designing the Sliding Surface

Consider the system

$$\dot{x} = f(x) + G(x)u$$

$$\left.\begin{array}{r} \operatorname{rank} G(x) = m \\ \text{satisfies controllability rank condition} \end{array}\right\} \text{around } x_0$$

Transform to **regular form**:

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_1(x_1, x_2) + G_2(x_1, x_2)u$$

$$x_1 \in R^{n-m}, x_2 \in R^m, \det G_2 \neq 0 \text{ around } x_0$$

Strategy:

1) choose $x_2 = -s_0(x_1)$ so that

$$\dot{x}_1 = f_1(x_1, -s_0(x_1))$$

has desired behavior,

2) choose u to enforce sliding on

$$s(x_1, x_2) = s_0(x_1) + x_2$$

Design dliding dynamics

Reaching

Example: Linear SISO Design ~ 1

$$\dot{x} = Ax + bu, \quad \operatorname{rank} b = 1, \quad (A, b) \text{ controllable}$$

reorder states to obtain

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \text{ with } b_2 \in R, \det b_2 \neq 0$$

transform to regular form:

$$z = Tx,\ T = \begin{bmatrix} I_{n-1} & -b_1 b_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix},\ T^{-1} = \begin{bmatrix} I_{n-1} & -b_1 \\ 0 & b_2 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix} u \quad \text{the pair } (A_{11}, A_{12}) \text{ is controllable}$$

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Example: Linear SISO Design ~ 2

To shape sliding dynamics, choose

$$z_2 = K z_1 \Rightarrow \dot{z}_1 = \left(A_{11} + A_{12} K\right) z_1, \quad z_1 \in R^{n-1}, z_2 \in R$$

Choose K by any means, pole placement, LQG, etc.

Now, $s(z) = -K z_1 + z_2$

To design the reaching control u take

$$V(z) = \frac{1}{2} \| s(z) \|_Q^2 = \frac{1}{2} s^T(z) Q\, s(z), \quad Q^T = Q > 0$$

$$\dot{V}(z) = s^T Q\, \dot{s} = z^T [\text{-}K \quad 1]^T Q\, [\text{-}K \quad 1] A z + s^T Q u$$

$$u_i = -\kappa_i(z) \operatorname{sgn} s_i^*, \quad s^*(z) = Q s(z),$$

$$\left\| \kappa(z) \right\| > \left\| z^T [\text{-}K \quad I_m]^T Q\, [\text{-}K \quad I_m] A z \right\|$$

Example: Linear MIMO Design ~ 1

$$\dot{x} = Ax + Bu, \quad \text{rank}\, B = m, \quad (A, B) \text{ controllable}$$

reorder states to obtain

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \text{ with } B_2 \in R^{m \times m}, \det B_2 \neq 0$$

transform to regular form:

$$z = Tx, T = \begin{bmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & B_2^{-1} \end{bmatrix}, T^{-1} = \begin{bmatrix} I_{n-m} & -B_1 \\ 0 & B_2 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ I \end{bmatrix} u \quad \text{the pair } (A_{11}, A_{12}) \text{ is controllable}$$

Example: Linear MIMO Design ~ 2

To shape sliding dynamics,

$$z_2 = K z_1 \Rightarrow \dot{z}_1 = \left(A_{11} + A_{12} K \right) z_1$$

Choose K by any means, pole placement, LQG, etc.

Now, $s(z) = -K z_1 + z_2$

To design the reaching control u take

$$V(z) = \frac{1}{2} \| s(z) \|_Q^2 = \frac{1}{2} s^T(z) Q\, s(z), \quad Q^T = Q > 0$$

$$\dot{V}(z) = s^T Q\, \dot{s} = z^T [-K \quad I_m]^T Q\, [-K \quad I_m] A z + s^T Q u$$

$$u_i = -\kappa_i(z) \operatorname{sgn} s_i^*, \quad s^*(z) = Q s(z),$$

$$\| \kappa(z) \| > \left\| z^T [-K \quad I_m]^T Q\, [-K \quad I_m] A z \right\|$$

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Example: Underwater Vehicle

$$m\ddot{x} + c\dot{x}|\dot{x}| = u, \quad m, c \text{ uncertain}, \quad u \in [-U, U]$$

$$\dot{x} = v$$

$$\dot{v} = -\frac{c}{m}v|v| + \frac{1}{m}u$$

1. Choose a sliding surface: $s = v + \lambda x.$

   Why? Because $s \equiv 0 \Rightarrow \dot{x} = -\lambda x \quad (v = -\lambda x$ stabilizes first eq)

2. Choose reaching control based on $V(x, v) = s^T s = s^2,$

$$\dot{V} = 2s\begin{bmatrix}\lambda & 1\end{bmatrix}\begin{bmatrix} v \\ -\frac{c}{m}v|v| \end{bmatrix} + 2s\frac{1}{m}u \Rightarrow u = \begin{cases} -U & s > 0 \\ U & s < 0 \end{cases}$$

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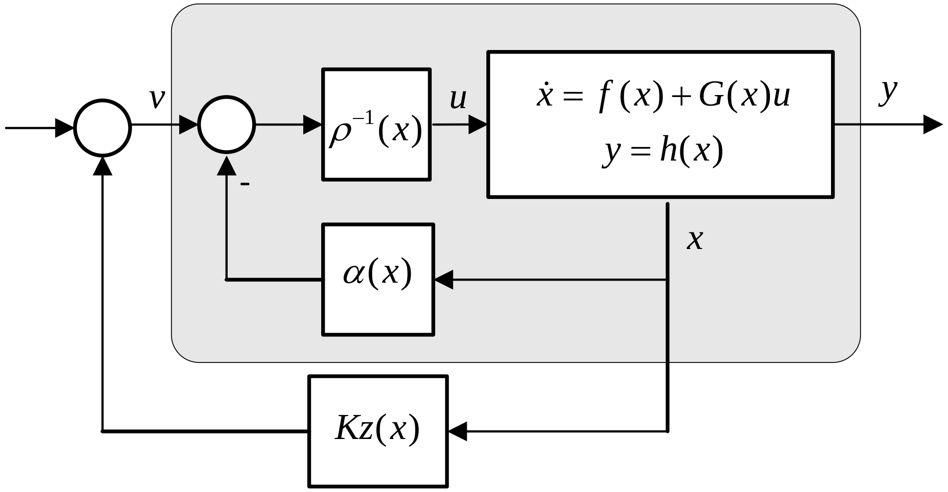
Control Based on Normal Form

$$\dot{\xi} = F(\xi, z, u)$$
$$\dot{z} = Az + E[\alpha(\xi, z) + \rho(\xi, z)u]$$
$$y = Cz$$

Recall Brunovsky structure of *A*,*E*



Choose $s(x)$ such that $s(x) = 0 \Leftrightarrow Kz(x) = 0$

$$u_i(x) = \begin{cases} u_i^+(x) & s_i(x) > 0 \\ u_i^-(x) & s_i(x) < 0 \end{cases}$$

Sliding Dynamics

$$s(x)=0 \Leftrightarrow Kz(x)=0$$

$$\Downarrow$$

$$K\dot{z}=KAz+KE\left[\alpha(x)+\rho(x)u_{eq}\right]=0$$

$$KE=I \qquad \Downarrow$$

Note : same as feedback linearizing control

$$u_{eq}=-\rho^{-1}(x)KAz(x)-\rho^{-1}(x)\alpha(x)$$

$$\Downarrow$$

$$\dot{z}=\left[I-EK\right]Az, \quad Kz(t_0)=0$$

Sliding Dynamics

Choosing *K*

One choice:

$$K = \begin{bmatrix} \overset{\leftarrow r_1 \rightarrow}{k_1} & 0 & \cdots & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & \cdots & 0 & \overset{\leftarrow r_m \rightarrow}{k_m} \end{bmatrix}, \quad k_i = \begin{bmatrix} a_{i,1} & \cdots & a_{i,r_i-1} & 1 \end{bmatrix}$$

Eigenvalues of (A+EK) are:

Sliding eigenvalues

$$m \text{ are } 0 \text{ and } r-m \text{ are } \quad \lambda\left(\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \\ -a_{i,1} & -a_{i,2} & \cdots & -a_{i,r_i-1} \end{bmatrix}\right), i = 1, \ldots, m$$

Reaching

Consider the positive definite quadratic form in s

$$V(x) = s^T Qs$$

Upon differentiation we obtain

$$\frac{d}{dt}V = 2\dot{s}^T Qs = 2[KAz + \alpha]^T QKz + 2u^T \rho^T QKz$$

If the controls are bounded, $|u|_i \le \overline{U}_i > 0$ $(0 > U_{\min,i} \le u_i \le U_{\max,i} > 0)$ then choose

$$u_i = \begin{cases} U_{\min,i} & s_i^*(x) > 0 \\ U_{\max,i} & s_i^*(x) < 0 \end{cases}, \quad i = 1, \ldots, m, \quad s^*(x) = \rho^T(x) QKz(x)$$

VSC Summary