# Hybrid System Control

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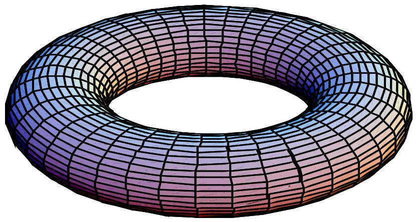
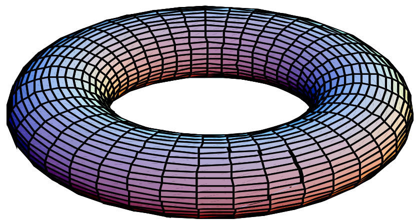


# Outline

- Hybrid System Models
  - Hybrid automaton
  - IP formulas
- Hybrid System Control
  - Inverted pendulum

# Hybrid System Models



# Hybrid Automaton

A hybrid automaton consists of:

(1) $Q$, discrete space (finite),

(2) $X$, continuous state space,

(3) $E$, set of transitions,

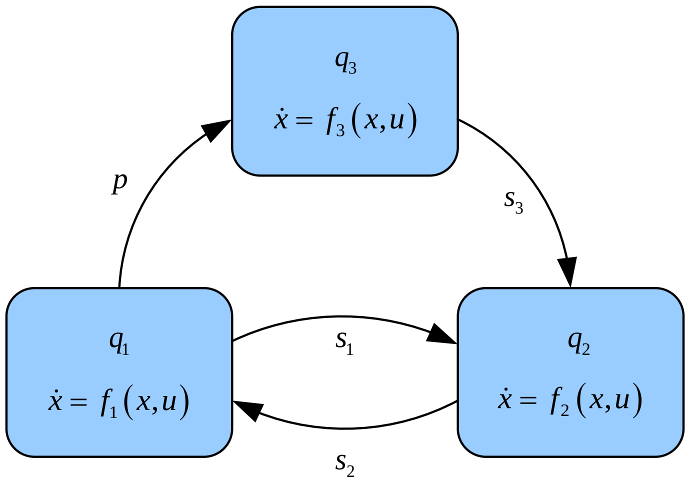
(4) $\Sigma$, event set,

(5) $G$, guard assignment function,

(6) $R$, state reset assignment function,

(7) $\mathfrak{L}$, logical specification,

(8) $F$, family of controlled vector fields.

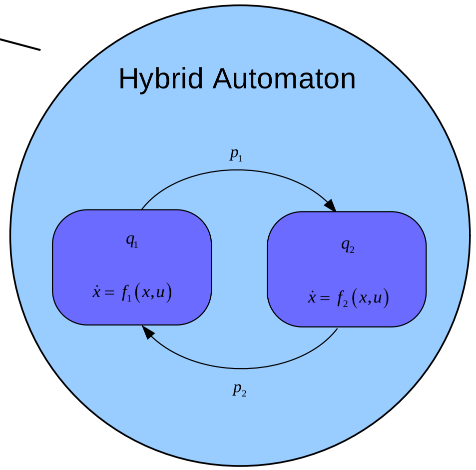



$$\mathfrak{L} = exactly\left(1,\left\{q_1(t),q_3(t),q_3(t)\right\}\right) \wedge exactly\left(1,\left\{q_1(t^+),q_3(t^+),q_3(t^+)\right\}\right) \wedge \left(q_1(t) \wedge s_1 \Rightarrow q_2(t^+)\right) \wedge \left(q_1(t) \wedge p \Rightarrow q_3(t^+)\right) \wedge \left(q_1(t) \wedge \neg(s_1 \vee p) \Rightarrow q_1(t^+)\right) \wedge \left(q_2(t) \wedge s_2 \Rightarrow q_1(t^+)\right) \wedge \left(q_2(t) \wedge \neg s_2 \Rightarrow q_2(t^+)\right) \wedge \left(q_3(t) \wedge s_3 \Rightarrow q_2(t^+)\right) \wedge \left(q_3(t) \wedge \neg s_3 \Rightarrow q_3(t^+)\right)$$

H. A. Henzinger, "The Theory of Hybrid Automata," 1996.

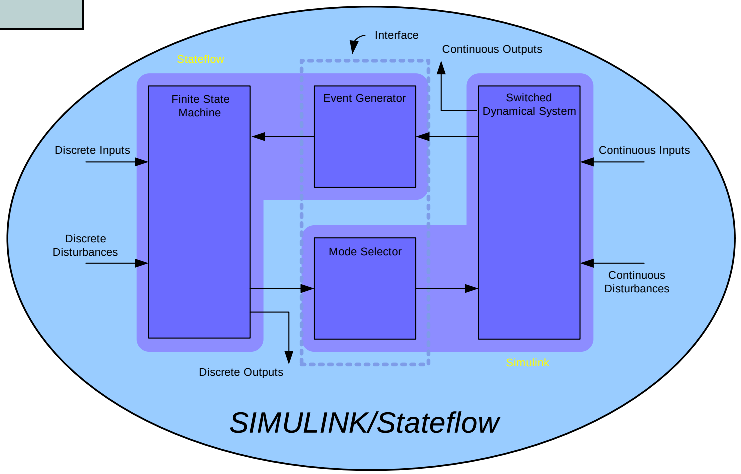
# Model Summary

Theory




Design

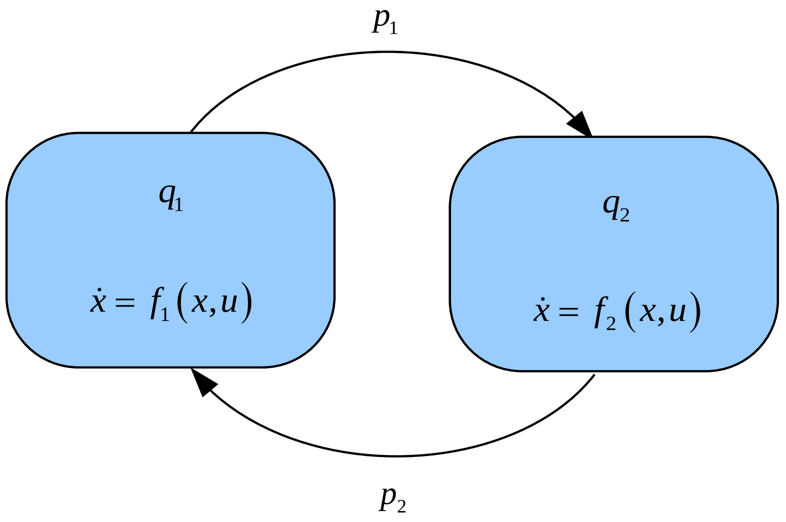
Closest to physics. Simulation




Dynamic Mixed Integer Program

$$\dot{x} = \delta_{q_1} f_1(x,u) + \delta_{q_2} f_2(x,u)$$
$$\delta_{p_1} + \delta_{q_1^+} - \delta_{q_1} \ge 0$$
$$1 - \delta_{q_1^+} - \delta_{q_2^+} \ge 0$$
$$-1 + \delta_{q_1^+} + \delta_{q_2^+} \ge 0$$
$$1 - \delta_{p_1} - \delta_{q_1} + \delta_{q_2^+} \ge 0$$
$$1 - \delta_{p_2} + \delta_{q_1^+} - \delta_{q_2} \ge 0$$
$$\delta_{p_2} + \delta_{q_2^+} - \delta_{q_2} \ge 0$$
$$0 \le \delta_{p_1} \le 1, 0 \le \delta_{p_2} \le 1, 0 \le \delta_{q_1} \le 1, 0 \le \delta_{q_2} \le 1$$
$$0 \le \delta_{q_1^+} \le 1, 0 \le \delta_{q_2^+} \le 1$$

# Logical Specification to IP Formulas, 1




$$
\begin{aligned}
L = (q_1 \oplus q_2) \wedge (q_1^+ \oplus q_2^+) \wedge \\
(q_1 \wedge p_1 \Rightarrow q_2^+) \wedge (q_1 \wedge \neg p_1 \Rightarrow q_1^+) \wedge \\
(q_2 \wedge p_2 \Rightarrow q_1^+) \wedge (q_2 \wedge \neg p_2 \Rightarrow q_2^+)
\end{aligned}
$$

# Logical Specification to IP Formulas, 2

$$\dot{x} = \delta_{q_1} f_1(x,u) + \delta_{q_2} f_2(x,u)$$

$$\delta_{p_1^+} + \delta_{q_1^+} - \delta_{q_1} \geq 0$$

$$1 - \delta_{q_1^+} - \delta_{q_2^+} \geq 0$$

$$-1 + \delta_{q_1^+} + \delta_{q_2^+} \geq 0$$

$$1 - \delta_{q_1} - \delta_{q_2} \geq 0$$

$$-1 + \delta_{q_1} + \delta_{q_2} \geq 0$$

$$1 - \delta_{p_1^+} - \delta_{q_1} + \delta_{q_2^+} \geq 0$$

$$1 - \delta_{p_2^+} + \delta_{q_1^+} - \delta_{q_2} \geq 0$$

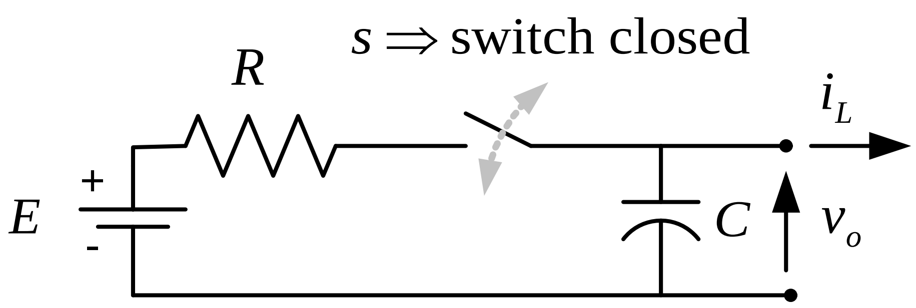
$$\delta_{p_2} + \delta_{q_2^+} - \delta_{q_2} \geq 0$$

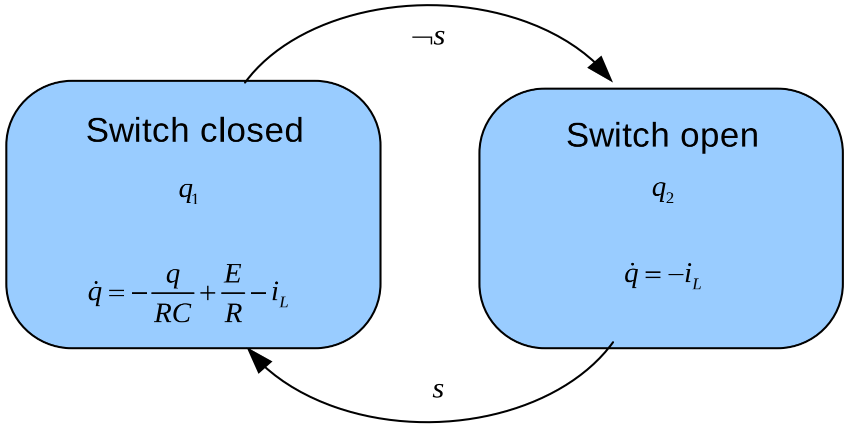
$$0 \leq \delta_{p_1} \leq 1, 0 \leq \delta_{p_2} \leq 1, 0 \leq \delta_{q_1} \leq 1, 0 \leq \delta_{q_2} \leq 1$$

$$0 \leq \delta_{q_1^+} \leq 1, 0 \leq \delta_{q_2^+} \leq 1$$

notation: for a proposition $\alpha$, $\delta_\alpha = 0,1 \Leftrightarrow \alpha =$ False, True

# Example: Power Conditioning System




$$L = (q_1 \oplus q_2) \wedge (q_1^+ \oplus q_2^+) \wedge$$
$$(q_1 \wedge \neg s \Rightarrow q_2^+) \wedge (q_1 \wedge s \Rightarrow q_1^+) \wedge$$
$$(q_2 \wedge s \Rightarrow q_1^+) \wedge (q_2 \wedge \neg s \Rightarrow q_2^+)$$

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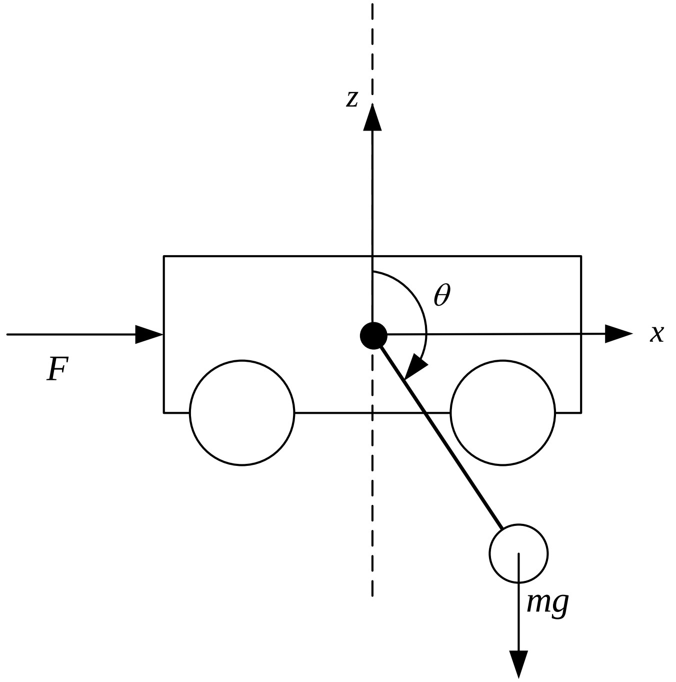
# Example: Power Conditioning System, 2

$$\dot{q} = \delta_{q_1}\left(-\frac{q}{RC}+\frac{E}{R}-i_L\right)+\delta_{q_2}\left(-i_L\right)$$

$$1-\delta_{q_1^+}-\delta_{q_2^+}\geq 0$$
$$-1+\delta_{q_1^+}+\delta_{q_2^+}\geq 0$$
$$1-\delta_{q_1}-\delta_{q_2}\geq 0$$
$$-1+\delta_{q_1}+\delta_{q_2}\geq 0$$
$$1-\delta_s-\delta_{q_1}+\delta_{q_2^+}\geq 0$$
$$1-\delta_s-\delta_{q_2}+\delta_{q_2^+}\geq 0$$
$$\delta_s-\delta_{q_1}+\delta_{q_1^+}\geq 0$$
$$\delta_s-\delta_{q_2}+\delta_{q_1^+}\geq 0$$
$$0\leq\delta_s\leq 1, 0\leq\delta_{q_1}\leq 1, 0\leq\delta_{q_2}\leq 1$$
$$0\leq\delta_{q_1^+}\leq 1, 0\leq\delta_{q_2^+}\leq 1$$

# Inverted Pendulum ~ 1




$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} M_c + m_p & l m_p \cos\theta \\ l m_p \cos\theta & l^2 m_p \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} - \begin{bmatrix} F + l m_p \omega^2 \sin\theta \\ g l m_p \sin\theta \end{bmatrix} = 0$$

$$\dot{x} = v$$

$$\dot{\theta} = \omega$$

$$2\dot{v} + \cos\theta\,\dot{\omega} = F + \omega^2 \sin\theta$$

$$\cos\theta\,\dot{v} + \dot{\omega} = \sin\theta$$

Suppose we choose $F$ to regulate $\dot{v}$:

$$F = \left(2 - \cos^2\theta\right)\dot{v} - \left(\omega^2 - \cos\theta\right)\sin\theta$$

So the pendulum equations become

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \sin\theta - u\cos\theta$$

where $u = \dot{v}$ is treated as a control.

# Inverted Pendulum ~ 2

Suppose we wish to design a global feedback controller that will steer any initial state to the upright position with the constraint $u \in [-1,1]$.

Feedback linearization will not work globally, choose $u$ so that

$$\ddot{\theta} - \sin\theta + u\cos\theta = 0 \rightarrow \ddot{\theta} + 2\dot{\theta} + \theta = 0$$

$\Rightarrow u = \dfrac{2\dot{\theta} + \theta + \sin\theta}{\cos\theta}$ works within constraints only, near $\theta = 0$.

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# Inverted Pendulum ~ 3 Swing-up Strategy

$$\dot{\theta} = \omega \qquad E_{pend} = \tfrac{1}{2}\omega^2 + (\cos\theta - 1)$$

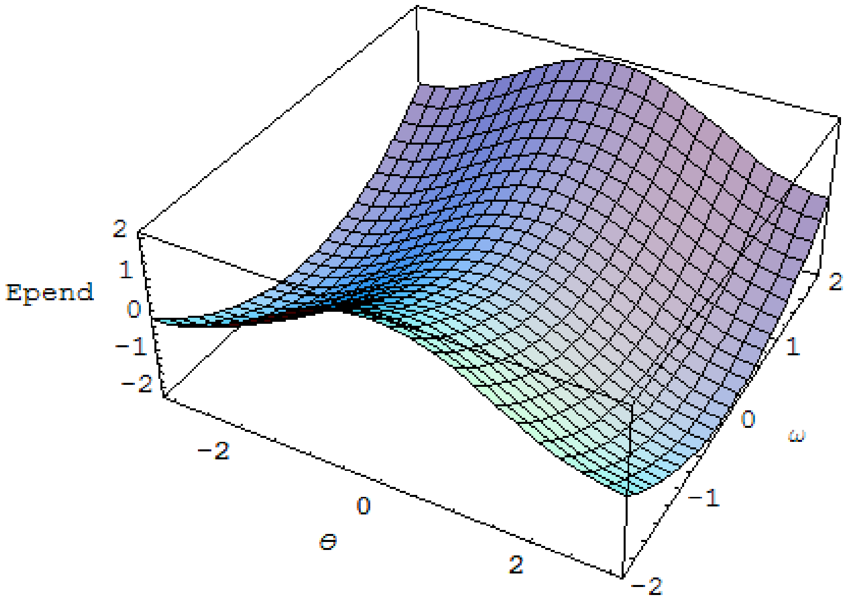
$$\dot{\omega} = \sin\theta - u\cos\theta \qquad E_{pend} = 0 \text{ when } \omega = 0, \theta = 0$$

1) pump/remove energy into system until $E_{pend} \approx 0,\ E_{pend} \in [-\varepsilon, \varepsilon]$

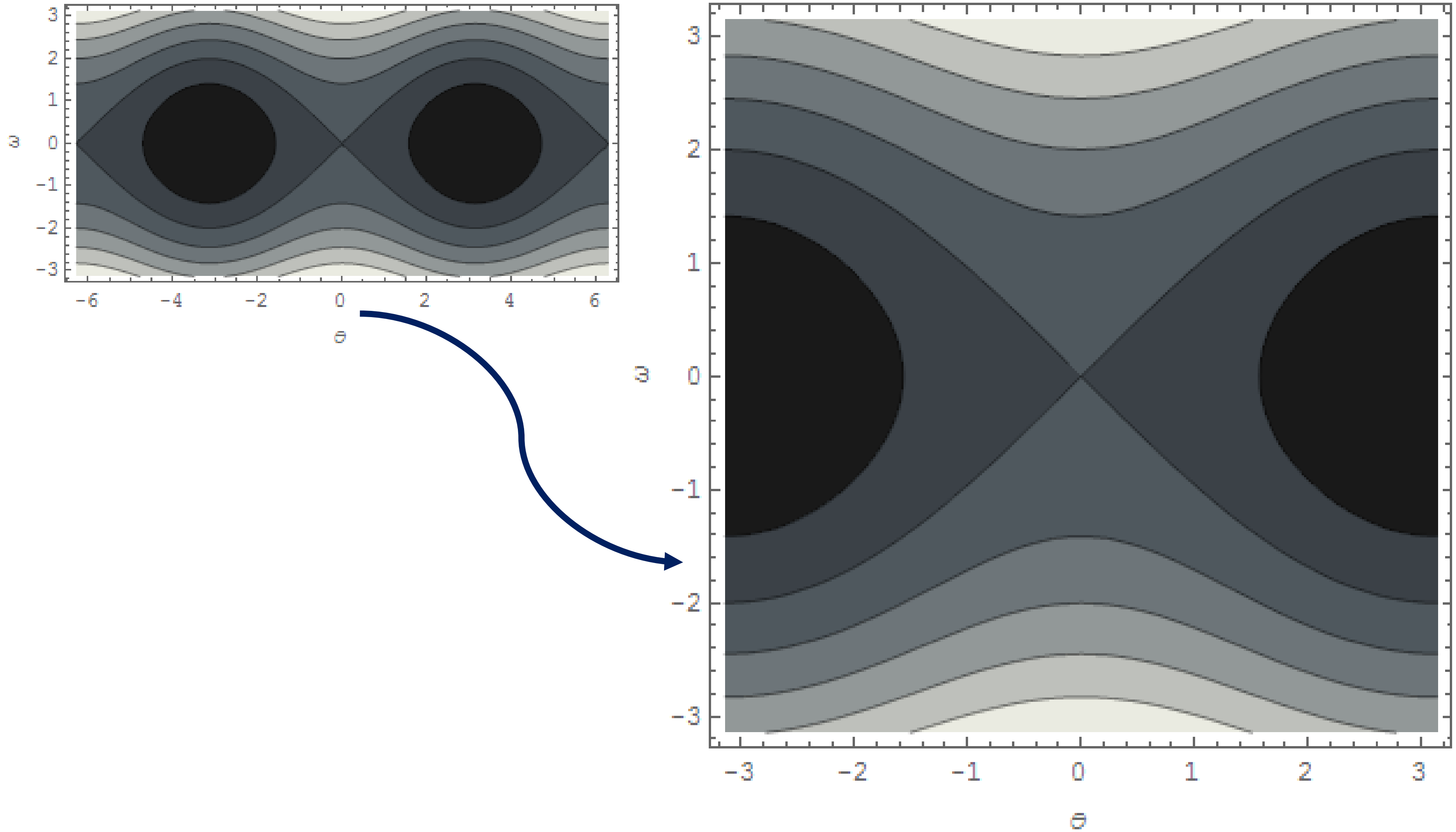
$$\dot{E} = \omega(\sin\theta - u\cos\theta) - \sin\theta\,\omega = -u\omega\cos\theta$$

2) wait until pendulum is close to upright

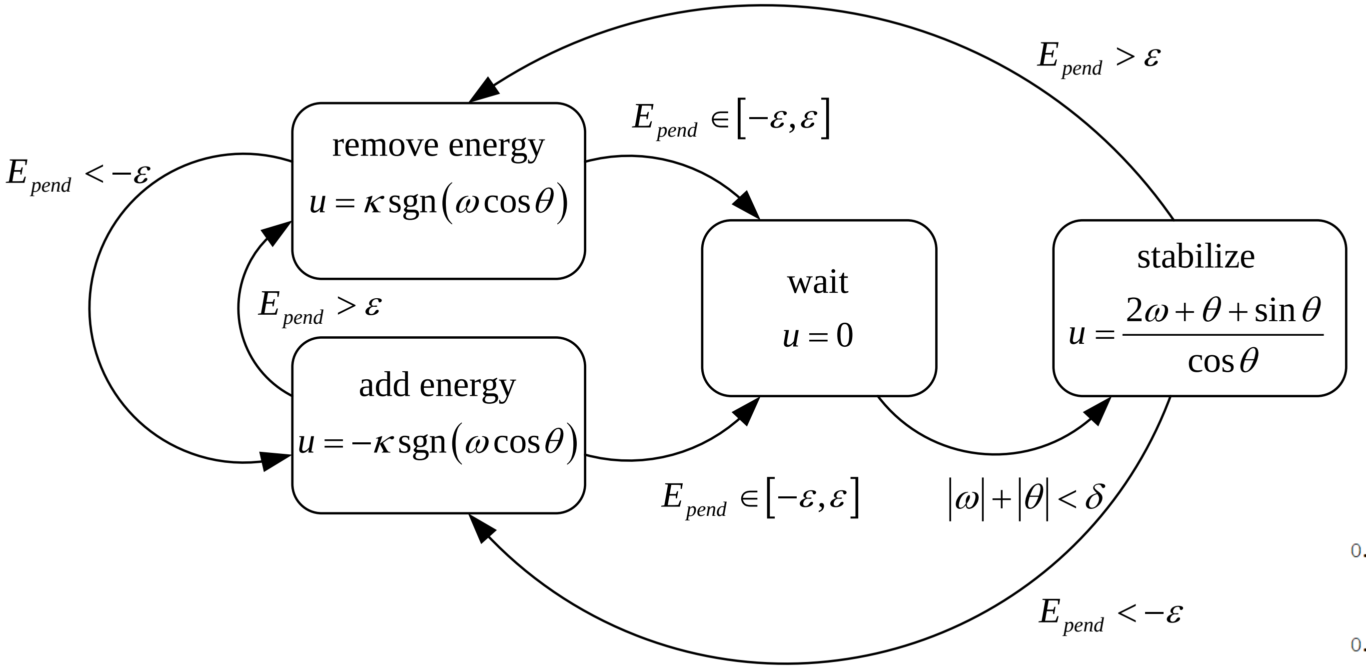
3) apply feedback linearizing control



# Inverted Pendulum ~ 4 Constant Energy Contours



# Inverted Pendulum ~ 5 Control Strategy




Constant energy trajectories, $u = 0$, in 'wait' state. Switch to 'stabilize' in blue box.