Optimal Control Systems

Introduction

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Outline

What is Optimal Control?

- Optimal control is an approach to control systems design that seeks the best possible control with respect to a performance metric.
- The theory of optimal control began to develop in the WW II years. The main result of this period was the Wiener-Kolmogorov theory that addresses linear SISO systems with Gaussian noise.
- A more general theory began to emerge in the 1950's and 60's
  - In 1957 Bellman published his book on Dynamic Programming
  - In 1960 Kalman published his multivariable generalization of Wiener-Kolmogorov
  - In 1962 Pontryagin et al published the maximal principle
  - In 1965 Isaacs published his book on differential games

Course Content

1. Introduction
   - Optimization basics
   - Intro to Variational Calculus
2. Variational Calculus and the Minimum Principle
   - Unconstrained Control problems
   - Control and State Constraints
3. Dynamic programming
   - Principle of Optimality
   - The Hamilton-Jacobi-Bellman Equation
4. Min-Max Optimal Control
   - Min-Max Control
   - Game Theory
5. Hybrid Systems
   - Hybrid Systems Basics
   - Hybrid Systems Optimal Control

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- S. J. Citron, Elements of Optimal Control. New York: Holt, Rinehart and Winston, Inc., 1969.
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Problem Definition

We will define the basic *optimal control problem*:

- Given system dynamics

$$\dot{x} = f(x, u), \quad x \in X \subset R^n, u \in U \subset R^m$$

- Find a control $u(t),\ t \in [0, T]$ that steers the system from an initial state $x(0) = x_0$ to a target set G and minimizes the cost

$$J(u(\cdot)) = g_T(x(T)) + \int_0^T g(x(t), u(t))\, dt$$

Remark

g_T is called the terminal cost and g is the running cost. The terminal time T can be fixed or free. The target set can be fixed or moving.

Open Loop vs. Closed Loop

- If we are concerned with a single specified initial state x_0, then we might seek the optimal control control $u(t)$, $u : R \to R^m$ that steers the system from the initial state to the target. This is an open loop control.
- On the other hand, we might seek the optimal control as a function of the state $u(x)$, $u : R^n \to R^m$. This is a closed loop control; sometimes called a synthesis.
- The open loop control is sometimes easier to compute, and the computations are sometimes performed online – a method known as model predictive control.
- The closed loop control has the important advantage that it is robust with respect to model uncertainty, and that once the (sometimes difficult) computations are performed off-line, the control is easily implemented online.

Example – a Minimum Time Problem

Consider steering a unit mass, with bounded applied control force, from an arbitrary initial position and velocity to rest at the origin in minimum time. Specifically,

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= u, \quad |u| \leq 1 \end{aligned}$$

The cost function is

$$J = \int_0^T dt \equiv T$$

Remark

This is an example of a problem with a control constraint.

Example – a Minimum Fuel Problem

Consider the decent of a moon lander.

$$\begin{aligned} \dot{h} &= v \\ \dot{v} &= -g + \frac{u}{m} \\ \dot{m} &= -ku \end{aligned}$$

The thrust u is used to steer the system to $h = 0, v = 0$. In addition we wish to minimize the fuel used during landing, i.e.

$$J = \int_0^t k u dt$$

Furthermore, u is constrained, $0 \leq u \leq c$, and the state constraint $h \geq 0$ must be respected.

Remark

This problem has both control and state constraints.

Example – a Linear Regulator Problem

Consider a system with linear dynamics

$$\dot{x} = Ax + Bu$$

We seek a *feedback* control that steers the system from an arbitrary initial state x_0 towards the origin in such a way as to minimize the cost

$$J = x^T(T)\,Q_T x(T) + \frac{1}{2T}\int_0^T \left\{ x^T(t)\,Qx(t) + u^T(t)\,Ru(t) \right\} dt$$

The final time T is considered fixed.

Example – a Robust Servo Problem

Consider a system with dynamics

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u\end{aligned}$$

where w is an external disturbance. The goal is to find an output (y) feedback synthesis such that the performance variables (process errors) z remain close to zero. Note that $w(t)$ can be characterized in several ways, stochastic (the H_2 problem)

$$J = E\left[\int_{-\infty}^{\infty} z^T(t)\, z(t)\, dt\right]$$

or deterministic (the H_∞ problem)

$$J = \max_{\|w\|_2 = 1} \int_{-\infty}^{\infty} z^T(t) z(t) dt$$

Definitions

Consider the scalar function

$$f(x), \quad x \in R^n$$

which is defined and smooth on a domain $D \subset R^n$. We further assume that the region D is defined by a scalar inequality $\psi(x) \le 0$, i.e.,

$$\begin{aligned} \mathrm{int}D &= \{x \in R^n \,|\psi(x) < 0\} \\ \partial D &= \{x \in R^n \,|\psi(x) = 0\} \end{aligned}$$

Definitions

Definition (Local Minima & Maxima)

An interior point $x^* \in \text{int}D$ is a local minimum if there exists a neighborhood U of x^* such that

$$f(x) \geq f(x^*) \quad \forall x \in U$$

It is a local maximum if

$$f(x) \leq f(x^*) \quad \forall x \in U$$

Similarly, for a point $x^* \in \partial D$, we use a neighborhood U of x^* within ∂D. With this modification boundary local minima and maxima are defined as above.

Optimal Interior Points

- Necessary conditions. A point $x^* \in \mathrm{int}D$ is an extremal point (minimum or maximum) only if
$$\frac{\partial f}{\partial x}(x^*) = 0$$
- Sufficient conditions. x^* is a minimum if
$$\frac{\partial^2 f}{\partial x^2}(x^*) > 0$$
a maximum if
$$\frac{\partial^2 f}{\partial x^2}(x^*) < 0$$

Optimization with constraints – Necessary Conditions

We need to find extremal points of $f(x)$, with $x \in \partial D$. i.e., find extremal points of $f(x)$ subject to the constraint $\psi(x) = 0$.

Consider a more general problem where there are m constraints, i.e., $\psi : R^n \to R^m$.

Let λ be an m-dimensional constant vector (called Lagrange multipliers) and define the function

$$H(x, \lambda) \triangleq f(x) + \lambda^T \psi(x)$$

Then x^* is an extremal point only if

$$\frac{\partial H(x^*, \lambda)}{\partial x} = 0, \quad \frac{\partial H(x^*, \lambda)}{\partial \lambda} \equiv \psi(x) = 0$$

Note there are $n + m$ equations in $n + m$ unknowns x, λ

Significance of the Lagrange Multiplier

Consider extremal points of $f(x_1, x_2)$ subject to the single constraint $\psi(x_1, x_2) = 0$. At an extremal point (x_1^*, x_2^*) we must have

$$df(x_1^*, x_2^*) = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} dx_1 + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} dx_2 = 0 \tag{1}$$

but dx_1 and dx_2 are not independent. They satisfy

$$d\psi(x_1^*, x_2^*) = \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_1} dx_1 + \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_2} dx_2 = 0 \tag{2}$$

From (1) and (2) it must be that

$$\frac{\partial f(x_1^*, x_2^*)/\partial x_1}{\partial \psi(x_1^*, x_2^*)/\partial x_1} = \frac{\partial f(x_1^*, x_2^*)/\partial x_2}{\partial \psi(x_1^*, x_2^*)/\partial x_2} \triangleq -\lambda$$

Accordingly, (1) and (2) yield

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \lambda \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_1} = 0, \quad \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} + \lambda \frac{\partial \psi(x_1^*, x_2^*)}{\partial x_2} = 0$$

Optimization with Constraints – Sufficient Conditions

$$df = \frac{\partial H}{\partial x}dx + \frac{1}{2}dx^T\frac{\partial^2 H}{\partial x^2}dx - \lambda^T d\psi + h.o.t.$$

but

$$d\psi = \frac{\partial \psi}{\partial x}dx = 0 \Rightarrow dx \in \ker\frac{\partial \psi}{\partial x} \Rightarrow dx = \Psi d\alpha$$

$$\Psi = \text{span}\ \ker\frac{\partial \psi(x)}{\partial x}$$

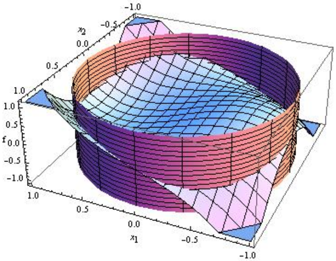
Thus, for x^* extremal $(\partial H/\partial x = 0, \psi = 0)$

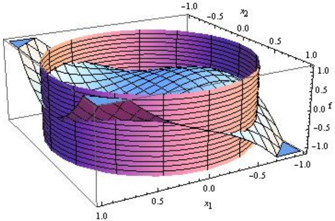
$$df(x^*) = \frac{1}{2}d\alpha^T\Psi^T\frac{\partial^2 H(x^*)}{\partial x^2}\Psi d\alpha + h.o.t.$$

$$\Psi^T\frac{\partial^2 H(x^*)}{\partial x^2}\Psi > 0 \Rightarrow \min, \quad \Psi^T\frac{\partial^2 H(x^*)}{\partial x^2}\Psi < 0 \Rightarrow \max$$

Example

$$f\left(x_1, x_2\right) = x_1\left(x_1^2 + 2x_2^2 - 1\right), \quad \psi\left(x_1, x_2\right) = x_1^2 + x_2^2 - 1$$





Interior:

$$(x_1, x_2, f) = (0, -0.707, 0) \vee (0, 0.707, 0) \vee (-0.577, 0, 0.385) \vee (0.577, 0. - 0.385)$$

Boundary:

$$(x_1, x_2, \lambda, f) = (-0.577, -0.8165, 1.155, -0.385) \vee (-0.577, 0.8165, 1.155, -0.385) \\ \vee (0.577, -0.8165, -1.155, 0.385) \vee (0.577, 0.8165, -1.155, 0.385) \\ \vee (-1, 0, 1, 0) \vee (1, 0, -1, 0)$$

Optimizing a Time Trajectory

- We are interested in steering a controllable system along a trajectory that is optimal in some sense.
- Three methods are commonly used to address such problems:
  - The ‘calculus of variations’
  - The Pontryagin ‘maximal Principle’
  - The ‘principle of optimality’ and dynamic programming
- The calculus of variations was first invented to characterize the dynamical behavior of physical systems governed by a conservation law.

Calculus of Variations: Lagrangian Systems

A Lagrangian System is characterized as follows:

- The system is define in terms of a vector of *configuration coordinates*, q, associated with velocities $\dot{q}$.
- The system has kinetic energy $T\left(\dot{q},q\right)=\dot{q}^{T}M\left(q\right)\dot{q}/2$, and potential energy $V\left(q\right)$ from which we define the Lagrangian

$$L\left(\dot{q},q\right)=T\left(\dot{q},q\right)-V\left(q\right)$$

- The system moves along a trajectory $q\left(t\right)$, between initial and final times t_1,t_2 in such a way as to minimize the integral

$$J\left(q\left(t\right)\right)=\int_{t_1}^{t_2}L\left(\dot{q}\left(t\right),q\left(t\right)\right)dt$$

Examples of Lagrangian Systems

$$T = \frac{1}{2}\dot{q}^T M(q)\,\dot{q}$$

$$M(q) = \begin{pmatrix} \ell_1^2(m_1+m_2) + \ell_2^2 m_2 + 2\ell_1\ell_2 m_2\cos\theta_2 & \ell_2 m_2(\ell_2+\ell_1\cos\theta_2) \\ \ell_2 m_2(\ell_2+\ell_1\cos\theta_2) & \ell_2^2 m_2 \end{pmatrix}$$

$$V(q) = m_1 g\,(g\ell_1(m_1+m_2)\sin\theta_1 + g\ell_2 m_2\sin(\theta_1+\theta_2))$$