Optimal Control Systems

Calculus of Variations

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Outline

Calculus of Variations: Lagrangian Systems

A Lagrangian System is characterized as follows:

- The system is define in terms of a vector of *configuration coordinates*, q, associated with velocities $\dot{q}$.
- The system has kinetic energy $T(\dot{q}, q) = \dot{q}^T M(q)\dot{q}/2$, and potential energy $V(q)$ from which we define the Lagrangian

$$L(\dot{q}, q) = T(\dot{q}, q) - V(q)$$

- The system moves along a trajectory $q(t)$, between initial and final times t_1, t_2 in such a way as to minimize the integral

$$J(q(t)) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t))\, dt$$

Necessary Conditions: Fixed Terminal Time –1

- A real-valued, continuously differentiable function $q(t)$ on the interval $[t_1, t_2]$ will be called *admissible*.
- Let $q^*(t)$ be an optimal admissible trajectory and $q(t,\varepsilon)$ a not necessarily optimal trajectory, with
$$q(t,\varepsilon) = q^*(t) + \varepsilon\eta(t)$$
where $\varepsilon > 0$ is a small parameter and $\eta(t)$ is arbitrary.
- Then
$$J(q(t,\varepsilon)) = \int_{t_1}^{t_2} L(\dot{q}^*(t) + \varepsilon\dot{\eta}(t), q^*(t) + \varepsilon\eta(t))\, dt$$
- An extremal of J is obtained from
$$\delta J(q(t)) = \left.\frac{\partial J(q(t,\varepsilon))}{\partial\varepsilon}\right|_{\varepsilon=0} = 0$$

Fixed Terminal Time – 2

$$\delta J\left(q\left(t\right)\right)=\int_{t_1}^{t_2}\left(\frac{\partial L\left(\dot{q}^*\left(t\right),q^*\left(t\right)\right)}{\partial\dot{q}^*}\dot{\eta}\left(t\right)+\frac{\partial L\left(\dot{q}^*\left(t\right),q^*\left(t\right)\right)}{q^*}\eta\left(t\right)\right)dt$$

We can apply the ‘integration by parts’ formula

$$\int udv=uv-\int vdu$$

to the first term to obtain

$$\delta J\left(q\left(t\right)\right)=\int_{t_1}^{t_2}\left(-\frac{d}{dt}\frac{\partial L(\dot{q}^*\left(t\right),q^*\left(t\right))}{\partial\dot{q}^*}+\frac{\partial L(\dot{q}^*\left(t\right),q^*\left(t\right))}{q^*}\right)\eta\left(t\right)\ dt$$
$$+\frac{\partial L(\dot{q}^*\left(t\right),q^*\left(t\right))}{\partial\dot{q}^*}\eta\left(t\right)\Big|_{t_1}^{t_2}$$

Fixed Terminal Time – 3

Now, set $\delta J\left(q\left(t\right)\right)=0$ to obtain:

- the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L\left(\dot{q}^{*}\left(t\right),q^{*}\left(t\right)\right)}{\partial\dot{q}^{*}}-\frac{\partial L\left(\dot{q}^{*}\left(t\right),q^{*}\left(t\right)\right)}{\partial q^{*}}=0$$

- the transversality conditions

$$\frac{\partial L\left(\dot{q}^{*}\left(t_{1}\right),q^{*}\left(t_{1}\right)\right)}{\partial\dot{q}^{*}}\delta q\left(t_{1}\right)=0,\quad\frac{\partial L\left(\dot{q}^{*}\left(t_{2}\right),q^{*}\left(t_{2}\right)\right)}{\partial\dot{q}^{*}}\delta q\left(t_{2}\right)=0$$

Remark

These results allow us to treat problems in which the initial and terminal times are fixed and individual components of $q(t_1)$ and $q(t_2)$ are fixed or free. Other cases of interest include: 1) the terminal time is free, and 2) the terminal time is related to the terminal configuration, e.g., by a relation $\varphi\left(q\left(t_{2}\right),t_{2}\right)=0$.

Necessary Conditions: Free Terminal Time – 1

Consider the case of fixed initial state and free terminal time. Let $q^*(t)$ be the optimal trajectory with optimal terminal time t_2^*. The perturbed trajectory terminates at time $t_2^* + \delta t_2$. Its end state is $q^*(t_2^*) + \varepsilon\eta(t_2^* + \varepsilon\tau)$. The perturbed cost is

$$J(q^*, \delta q, \delta t) = \int_{t_1}^{t_2^* + \delta t} L(\dot{q}^*(t) + \delta\dot{q}(t), q^*(t) + \delta q(t))\, dt$$

From this we obtain:

$$\delta J = \int_{t_1}^{t_2^*} \left(-\frac{d}{dt} \frac{\partial L(\dot{q}^*(t), q^*(t))}{\partial \dot{q}^*} + \frac{\partial L(\dot{q}^*(t), q^*(t))}{q^*} \right) \delta q(t)\, dt - \frac{\partial L(\dot{q}^*(t_1^*), q^*(t_1^*))}{\partial \dot{q}^*} \delta q(t_1^*)$$
$$+ \frac{\partial L(\dot{q}^*(t_2^*), q^*(t_2^*))}{\partial \dot{q}^*} \delta q(t_2^*) + L(\dot{q}^*(t_2^* + \delta t), q^*(t_2^*))\, \delta t$$

Now, we want to allow both the final time and the end point to vary. The actual end state is:

$$\delta q_2 \triangleq \delta q(t_2^* + \delta t) = \delta q(t_2^*) + \dot{q}^*(t_2^*)\, \delta t$$

Free Terminal Time – 2

$$\delta J = \int_{t_1}^{t_2^*} \left(-\frac{d}{dt}\frac{\partial L\left(\dot{q}^*(t), q^*(t)\right)}{\partial \dot{q}^*} + \frac{\partial L\left(\dot{q}^*(t), q^*(t)\right)}{q^*} \right) \delta q(t)\, dt - \frac{\partial L\left(\dot{q}^*(t_1^*), q^*(t_1^*)\right)}{\partial \dot{q}^*} \delta q(t_1^*)$$
$$+ \frac{\partial L\left(\dot{q}^*(t_2^*), q^*(t_2^*)\right)}{\partial \dot{q}^*} \delta q(t_2^*) + \left(L\left(\dot{q}^*(t_2^*), q^*(t_2^*)\right) - \frac{\partial L\left(\dot{q}^*(t_2^*), q^*(t_2^*)\right)}{\partial \dot{q}^*} \dot{q}^*(t_2^*) \right) \delta t$$

Thus, we have have the Euler-Lagrange Equations, as before, but the transversality conditions become:

- the previous conditions:

$$\frac{\partial L\left(\dot{q}^*(t_1), q^*(t_1)\right)}{\partial \dot{q}^*} \delta q(t_1) = 0, \quad \frac{\partial L\left(\dot{q}^*(t_2^*), q^*(t_2^*)\right)}{\partial \dot{q}^*} \delta q(t_2^*) = 0$$

- plus, additional condition:

$$\left(L\left(\dot{q}^*(t_2^*), q^*(t_2^*)\right) - \frac{\partial L\left(\dot{q}^*(t_2^*), q^*(t_2^*)\right)}{\partial \dot{q}^*} \dot{q}^*(t_2^*) \right) \delta t = 0$$

Free Terminal Time – 3

Ordinarily, the initial state and time are fixed so that the transversality conditions require

$$\delta q\left(t_1\right)=0$$

For free terminal time δt is arbitrary. Thus from the transversality conditions:

- If the terminal state is fixed, $\delta q\left(t_2\right)=0$ the terminal condition is:

$$L\left(\dot{q}^*\left(t_2^*\right),q^*\left(t_2^*\right)\right)-\frac{\partial L\left(\dot{q}^*\left(t_2^*\right),q^*\left(t_2^*\right)\right)}{\partial\dot{q}^*}\dot{q}^*\left(t_2^*\right)=0$$

- If the terminal state is free, δq_2 arbitrary the terminal condition is:

$$\frac{\partial L\left(\dot{q}^*\left(t_2^*\right),q^*\left(t_2^*\right)\right)}{\partial\dot{q}^*}=0,\ L\left(\dot{q}^*\left(t_2^*\right),q^*\left(t_2^*\right)\right)=0$$

Control Example

First note that in the elementary variational calculus, we could simply replace q by x, and add the definition $\dot{x} = u$, so that the cost function becomes

$$J\left(x\left(t\right)\right) = \int_{t_1}^{t_2} L\left(u\left(t\right), x\left(t\right)\right) dt$$

Hence, we have a simple control problem. We consider 3 variants:

1. Free endpoint, fixed terminal time
2. Fixed endpoint, fixed terminal time
3. Fixed endpoint, free terminal time

Example 1: Free endpoint, Fixed terminal time

Suppose $x \in R$, $t_1 = 0$, $t_2 = 1$, and $L = \left(x^2 + u^2\right)/2$:

$$J\left(x\left(t\right)\right) = \int_0^1 \frac{1}{2}\left(x^2 + u^2\right) dt$$

The Euler-Lagrange equations become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \dot{u} - x = 0$$

The transversality conditions are:

$$x\left(0\right) = x_0, \ \frac{\partial L\left(x\left(1\right), u\left(1\right)\right)}{\partial u} = 0$$

Example 1, Cont'd

Thus, we have the equations

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$$x(0) = x_0, \ \frac{\partial L(x(1), u(1))}{\partial u} = 0 \Rightarrow u(1) = 0$$

Consequently,

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = e^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} t} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} a\cosh(t) + b\sinh(t) \\ b\cosh(t) + a\sinh(t) \end{bmatrix}$$

$$a = x_0, \ b = \frac{x_0 - e^2 x_0}{1 + e^2}$$

Example 1: Fixed endpoint, Fixed terminal time

Suppose we consider a fixed end point, say $x(t_2) = 0$ with the terminal time still fixed at $t_2 = 1$. Then the Euler-Lagrange equations remain the same, but the boundary conditions change to:

$$x(0) = x_0, \; x(1) = 0$$

Thus, we compute

$$a = x_0, \; b = -\frac{x_0 + e^2 x_0}{-1 + e^2}$$

Example 1: Fixed endpoint, Free terminal time

Suppose we consider a fixed end point, say $x(t_2) = x_f$ with the terminal time t_2 free. Then the Euler-Lagrange equations remain the same, but the boundary conditions change to:

$$x(0) = x_0, x(t_2) = x_f, \; (L - L_u u)\big|_{t_2} = 0$$

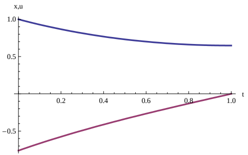
From this we compute

$$a = x_0, a^2 = b^2, a\cosh(t_2) + b\sinh(t_2) = x_f$$

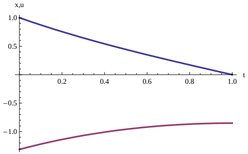
EXAMPLE 1, CONT'D

The figures below show the optimal control and state trajectories from the initial state $x_0 = 1$,$x_f = 0$, with $t_2 = 1$ for the fixed time case. For the free time case $x_f = 0.01$.

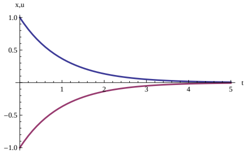
Free terminal state, Fixed time:



Fixed terminal state, Fixed Time:



Fixed terminal state, Free time:



REMARK

In the free time case, with $x_f = 0.01$, the final time is $t_2 = 4.60517$. With $x_f = 0$ the final time is $t_2 = \infty$. Note that the case of free terminal state and free terminal time (not shown) is trivial with $t_2 = 0$.

Variational Calculus with Differential Constraints

Systems with ‘nonintegrable’ differential , i.e., nonholonomic, constraints have been treated by variational methods. As before, we seek extremals of the functional:

$$J\left(q\left(t\right)\right)=\int_{t_1}^{t_2}L\left(\dot{q}\left(t\right),q\left(t\right),t\right)dt$$

But now subject to the constraints:

$$\varphi\left(\dot{q}\left(t\right),q\left(t\right),t\right)=0$$

where $\varphi : R^n \times R^n \times R \rightarrow R^m$.

Differential Constraints, Cont'd

The constraints are enforce for all time, so introduce the m Lagrange multipliers $\lambda(t)$ and consider the modified functional

$$J(q(t)) = \int_{t_1}^{t_2} L(\dot{q}(t), q(t), t) + \lambda^T(t)\varphi(\dot{q}(t), q(t), t)\, dt$$

We allow variations in both q and λ and t_2, Taking the variation and integrating by parts yields

$$\begin{aligned} \delta J = & \int_{t_1}^{t_2} \left(\left\{-\tfrac{d}{dt}\left[L_{\dot{q}} + \lambda^T\varphi_{\dot{q}}\right] + \left[L_q + \lambda^T\varphi_q\right]\right\}\delta q(t) + \varphi^T\delta\lambda\right) dt \\ & + \left[L_{\dot{q}} + \lambda^T\varphi_{\dot{q}}\right]_{t_2}\delta q_2 + \left(\left[L + \lambda^T\varphi_q\right] - \left[L_{\dot{q}} + \lambda^T\varphi_{\dot{q}}\right]\dot{q}\right)_{t_2}\delta t_2 \end{aligned}$$

Necessary Conditions with Constraints

Again we have the Euler-Lagrange equations, now in the form:

$$\frac{d}{dt}\left[L_{\dot{q}} + \lambda^T \varphi_{\dot{q}}\right] - \left[L_q + \lambda^T \varphi_q\right] = 0$$

The differential constraints:

$$\varphi\left(\dot{q}, q, t\right) = 0$$

and the boundary conditions

$$\left[L_{\dot{q}} + \lambda^T \varphi_{\dot{q}}\right]_{t_2} \delta q_2 = 0$$

With free terminal time, we also have

$$\left(\left[L + \lambda^T \varphi_q\right] - \left[L_{\dot{q}} + \lambda^T \varphi_{\dot{q}}\right] \dot{q}\right)_{t_2} = 0$$

EXAMPLE

Once again consider the problem

$$\dot{x} = u, \quad J(x(t)) = \int_0^{t_2} \frac{1}{2}\left(x^2 + u^2\right) dt$$

But now, add the constraint

$$\varphi(\dot{x}, x) = \dot{x}^2 - 1 \equiv u^2 - 1 = 0 \rightarrow u = \pm 1$$

The Euler-Lagrange equations now become

$$\frac{d}{dt}\left[L_u + \lambda^T \varphi_u\right] - \left[L_x + \lambda^T \varphi_x\right] = 0 \Rightarrow \dot{\lambda} = \frac{x}{2u}$$

REMARK

- *Note that* $u = \pm 1 \rightarrow \dot{u} = 0$ *almost everywhere,*
- *Also,* $\int_0^t u^2 dt = t$

Example Continued

We now consider three cases:

- free end point, fixed terminal time
- fixed end point, fixed terminal time
- fixed end point, free terminal time

Example: Free End Point, Fixed terminal Time

Steer from $x(0) = x_0$.

- system $\dot{x} = u$
- Euler equation $\dot{\lambda} = \frac{x}{2u}$
- constraint $u = \pm 1$
- initial condition $x(0) = x_0$
- terminal condition $\left[L_u + \lambda^T \varphi_u\right]_{t_2} = 0 \Rightarrow \lambda(1) = -\frac{1}{2}$

Example: Free End Point, Fixed terminal Time–2

The control is piecewise constant, switching between $u = +1$ and $u = -1$. Assume:

- there is only one switch that takes place at $t = T$ with $0 < T < 1$ with $u(0) = u_0 = \pm 1$ and $u(t) = -u_0, t > T$
- $\lambda(0) = \lambda_0$

$$x(t) = \begin{cases} x_0 + u_0 t & 0 < t < T \\ x_0 + u_0 T - u_0 (t - T) & T < t \leq 1 \end{cases}$$

$$\lambda(t) = \begin{cases} \lambda_0 + \frac{1}{2u_0}\left(x_0 + \frac{u_0}{2} t^2\right) & 0 < t < T \\ \lambda_0 + \frac{1}{2u_0}\left(x_0 + \frac{u_0}{2} T^2\right) - \left(\frac{x_0}{2u_0} + \frac{1}{2} T\right)(t - T) + \frac{1}{4}(t - T)^2 & T < t \leq 1 \end{cases}$$

$$\lambda(1) = -\frac{1}{2} \Rightarrow \lambda_0 + \frac{3}{4} + \left(\frac{x_0}{2u_0} - 1\right) T + T^2 = 0$$

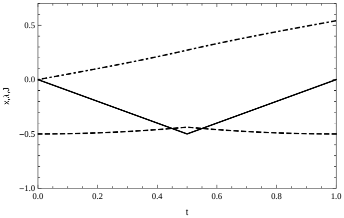
Example: Free End Point, Fixed terminal Time–3

Consider the case $x(0) = 0$. The necessary conditions are satisfied with

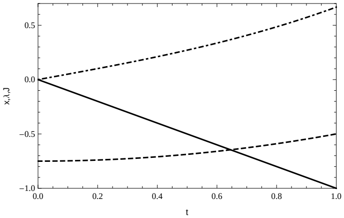
$$(u_0 = \pm 1) \wedge \left(-\frac{3}{4} \leq \lambda_0 \leq -\frac{1}{2}\right) \wedge \left(T = \frac{1}{2} \pm \frac{1}{2}\sqrt{-2 - 4\lambda_0}\right)$$

Two examples are given below. Only the case $\lambda_0 = -\frac{1}{2}, T = \frac{1}{2}$ is optimal.

$\lambda_0 = -\frac{1}{2}$:



$\lambda_0 = -\frac{3}{4}$:



Example: Fixed End Point, Fixed terminal Time

Steer from $x(0) = x_0$ to $x(1) = x_1$.

- system $\dot{x} = u$
- Euler equation $\dot{\lambda} = \frac{x}{2u}$
- constraint $u = \pm 1$
- boundary conditions $x(0) = x_0, x(1) = x_1$

Assume single switch at time T

$$\begin{aligned} & x_0 + u_0 T - u_0(1-T) = x_1 \wedge (u_0 = \pm 1) \wedge 0 \le T \le 1 \\ & \Rightarrow -1 + x_1 \le x_0 \le 1 + x_1 \wedge (u_0 = \pm 1) \wedge T = \frac{u_0 - x_0 + x_1}{2u_0} \end{aligned}$$

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Example: Free terminal Time, Fixed End Point

Steer from $x(0) = x_0$ to $x(t_2) = 0$

- system $\dot{x} = u$
- Euler equation $\dot{\lambda} = \frac{x}{2u}$
- constraint $u = \pm 1$
- initial condition $x(0) = x_0, x(t_2) = 0$
- terminal time condition $(L + \lambda^T \varphi_x) - (L_u - \lambda^T \varphi_u) = 0 \Rightarrow \lambda(t_2) = \frac{1}{4k} + \frac{1}{2}$

$$(x_0 \geq 0, t_2 \geq x_0) \wedge (u_0 = \pm 1) \wedge \left(T = \frac{t_2 - u_0 x_0}{2}\right) \wedge \left(\lambda_0 = \frac{(t_2 - 2T)^2 - 1}{4}\right)$$
$$\vee$$
$$(x_0 \leq 0, t_2 \geq -x_0) \wedge (u_0 = \pm 1) \wedge \left(T = \frac{t_2 - u_0 x_0}{2}\right) \wedge \left(\lambda_0 = \frac{(t_2 - 2T)^2 - 1}{4}\right)$$