Optimal Control Systems

Variational Calculus & The Minimum Principle

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OUTLINE

THE OPTIMAL CONTROL PROBLEM

We consider the system dynamics

$$\dot{x} = f(x, u), \quad x \in R^n, u \in U \subset R^m$$

and cost function

$$J(u(\cdot)) = \ell(x(t_2)) + \int_{t_1}^{t_2} L(x(t), u(t), t)\, dt$$

with the following terminal conditions:

- fixed initial state $x(t_1) = x_0$
- terminal time t_2 is free or fixed
- terminal state $x(t_2)$ is free or fixed by element

The Variation

Define the Lagrange multipliers $\lambda(t)$ and consider

$$J(u(\cdot)) = \ell(x(t_2)) + \int_{t_1}^{t_2} \{L(x(t), u(t), t) + \lambda^T(t)(f(x(t), u(t)) - \dot{x}(t))\} dt$$

We allow variations in u, x, λ, t_2 and we assume that these variations are completely arbitrary, other than the relationship implied by the system dynamics. There are no other constraints on x or u. Thus, we obtain

$$\begin{aligned} \delta J = {} & \ell_x \delta x(t_2) \\ & + \int_{t_1}^{t_2} \left[L_x \delta x + L_u \delta u + (f - \dot{x})^T \delta\lambda + \lambda^T (f_x \delta x + f_u \delta u - \delta\dot{x})\right] dt \\ & + \left[L + \lambda^T (f - \dot{x})\right]_{t_2} \delta t_2 \end{aligned}$$

The Hamiltonian

Define the Hamiltonian

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u)$$

so that the δJ can be written

$$\begin{aligned}\delta J = {} & \ell_x \delta x(t_2) + \left[L + \lambda^T (f - \dot{x})\right]_{t_2} \delta t_2 \\ & + \int_{t_1}^{t_2} \left[H_x \delta x + H_u \delta u + (f - \dot{x})^T \delta\lambda + \lambda^T \delta\dot{x}\right] dt\end{aligned}$$

Now apply integration by parts

$$\int_{t_1}^{t_2} \lambda^T(t)\, \delta\dot{x}\, dt = -\lambda^T(t_2)\left(\delta x(t_2) - \dot{x}(t_2)\,\delta t_2\right) + \int_{t_1}^{t_2} \dot{\lambda}^T(t)\, \delta x\, dt$$

so that

$$\begin{aligned}\delta J = {} & \left(\ell_x - \lambda^T\right)_{t_2} \delta x(t_2) + \left[L + \lambda^T (f - \dot{x}) + \lambda^T \dot{x}\right]_{t_2} \delta t_2 \\ & + \int_{t_1}^{t_2} \left[\left(H_x + \dot{\lambda}^T\right) \delta x + H_u \delta u + (f - \dot{x})^T \delta\lambda\right] dt\end{aligned}$$

Necessary Conditions without Control or State Constraints

Since we require $\delta J = 0$ for arbitrary variations we obtain the key necessary conditions for an optimal trajectory:

$$
\begin{aligned}
\dot{x}^* &= H_\lambda^T\left(x^*, \lambda^*, u^*\right) && \text{state equations} \\
\dot{\lambda}^{*T} &= -H_x\left(x^*, \lambda^*, u^*\right) && \text{adjoint equations} \\
H_u &= 0 \\
H\left(x^*, \lambda^*, u^*\right) &= c
\end{aligned}
$$

with the boundary conditions:

- initial state $x\left(t_1\right) = x_0$
- transversality condition $\left(\ell_x - \lambda^T\right)_{t_2} \delta x\left(t_2\right) = 0$
- if the terminal time is free, then $\left[L + \lambda f\right]_{t_2} = H_{t_2} = 0$

Remarks

Note that the transversality condition

$$\left(\ell_x - \lambda^T\right)_{t_2} \delta x\left(t_2\right) = 0$$

leads to different boundary conditions depending on the situation.

- if the final state (or component) is fixed, $x\left(t_2\right) = x_f$
- if the final state (or component) is free, then $\lambda\left(t_2\right) = \left.\frac{\partial \ell}{\partial x}\right|_{t_2}$
- suppose that the terminal condition is given as a constraint $\psi\left(x\left(t_2\right)\right) = 0$. This implies

$$\frac{\partial \psi(x)}{\partial x}\delta x = 0 \Rightarrow \delta x = \Psi \delta\alpha, \quad \operatorname{span}\Psi = \ker \frac{\partial \psi(x)}{\partial x}$$

so that if $x\left(t_2\right)$ is otherwise unconstrained

$$\left[\left(\ell_x - \lambda^T\right)\Psi\right]_{t_2} = 0$$

Remarks Cont'd

- It is easy to show that $H\left(x^*, \lambda^*, u^*\right) = c$

$$\begin{aligned} \frac{d}{dt}H\left(x^*, \lambda^*, u^*\right) = {} & H_x\left(x^*, \lambda^*, u^*\right)\dot{x} + H_\lambda\left(x^*, \lambda^*, u^*\right)\dot{\lambda} \\ & + H_u\left(x^*, \lambda^*, u^*\right)\dot{u} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}H\left(x^*, \lambda^*, u^*\right) = {} & H_x\left(x^*, \lambda^*, u^*\right)H_\lambda^T\left(x^*, \lambda^*, u^*\right) \\ & - H_x\left(x^*, \lambda^*, u^*\right)H_\lambda^T\left(x^*, \lambda^*, u^*\right) \\ = {} & 0 \end{aligned}$$

Example - Fixed Time to Target

Consider a problem with dynamics

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$

and cost

$$J = \frac{1}{2}\int_0^2 u^2\,dt$$

We wish to steer the system from an arbitrary initial state to the origin in specified time $t_2 = 2$ in such a way as to minimize cost. We compute

$$H = \tfrac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2\left(-x_2 + u\right)$$
$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 + \lambda_2$$

Example - Cont'd

The optimal control is given by

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = -\lambda_2$$

Thus, we need to solve

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -x_1 - \lambda_2$$
$$\dot{\lambda}_1 = 0, \ \dot{\lambda}_2 = -\lambda_1 + \lambda_2$$

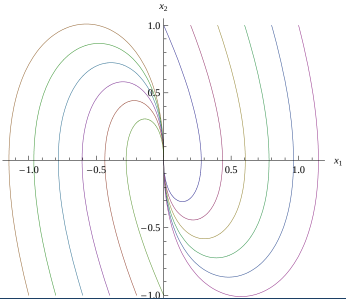
subject to the boundary conditions

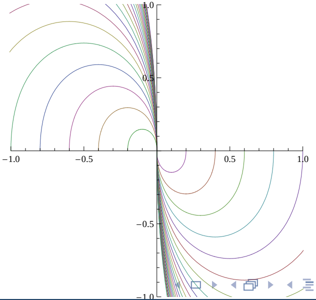
$$x_1(0) = a, \ x_2(0) = b, \ x_1(2) = 0, \ x_2(2) = 0$$

Example - Cont'd

$$x_1[t] \to \frac{e^{-t}\left(\begin{array}{l} a\left(-1+e^2\right)\left(-e^2+e^{2t}-e^t\left(-3+e^2(-1+t)+t\right)\right) \\ +b\left(e^{2t}\left(-3+e^2\right)-e^2\left(1+e^2\right)-e^t\left(-3+e^4(-1+t)+t-2e^2t\right)\right)\end{array}\right)}{4\left(-1+e^2\right)}$$

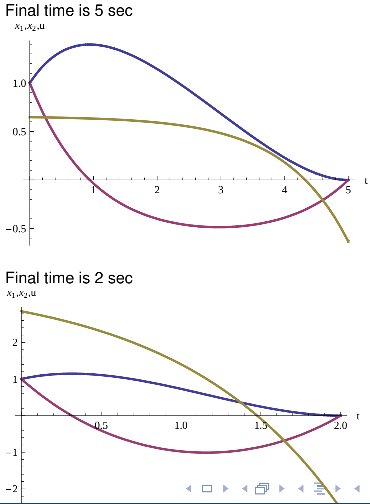
$$x_2[t] \to \frac{e^{-t}\left(-e^2+e^t\right)\left(be^t\left(-3+e^2\right)-b\left(1+e^2\right)+a\left(-1+e^2\right)\left(-1+e^t\right)\right)}{4\left(-1+e^2\right)}$$





Example - Cont'd

- The figures on the right illustrate how terminal time affects the optimal trajectories.
- The terminal time can be reduced arbitrarily, leading to large control peak magnitude.



EXAMPLE: FREE TIME TO TARGET

Once again consider a system with dynamics

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$

We wish to steer the system from the initial state $x_1(0) = a,\ x_2(0) = b$ to the origin with cost function

$$J = \int_0^{t_2} \left(1 + \tfrac{1}{2}u^2\right) dt$$

where the terminal time t_2 is free. Note that the cost is a combination of 'time to target' and control effort. The Hamiltonian is

$$H = 1 + \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 \left(-x_2 + u\right)$$

and the adjoint dynamics are

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1 + \lambda_2$$

Example - Cont'd

As before

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = -\lambda_2$$

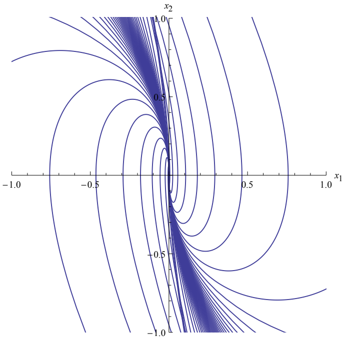
So the optimal system is described by the differential equations

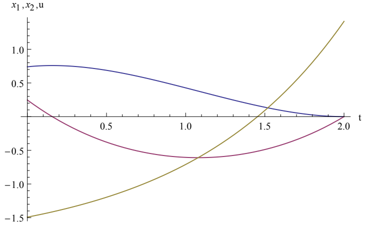
$$\dot{x}_1 = x_2, \; \dot{x}_2 = -x_1 - \lambda_2$$
$$\dot{\lambda}_1 = 0, \; \dot{\lambda}_2 = -\lambda_1 + \lambda_2$$

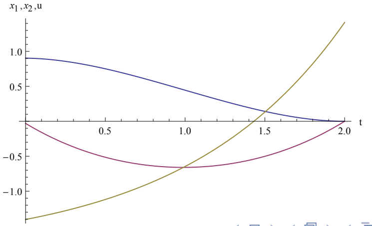
Now, let us consider the terminal conditions:

- initial state: $x_1(0) = a, \; x_2(0) = b$
- terminal state: $x_1(t_2) = 0, \; x_2(t_2) = 0$
- free terminal time: $H^*_{t_2} = 1 - \frac{1}{2}\lambda_2^2(t_2) = 0 \Rightarrow \lambda_2(t_2) = \pm\sqrt{2}$

Example - Cont'd







Synthesis by Backing out of the Target

Suppose our goal is to steer a system from an initial state to a target set S defined by $\psi(x) \le 0$, with cost function J and free terminal time. Our necessary conditions include:

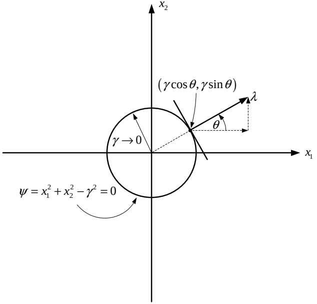
- state and adjoint equations
- control $u^*(x, \lambda)$
- terminal state belongs to the target set boundary
- transversality condition at terminal time, $\left(\ell_x - \lambda^T\right)_{t_2} \delta x(t_2) = 0$
- free terminal time condition, $H^*_{t_2} = 0$

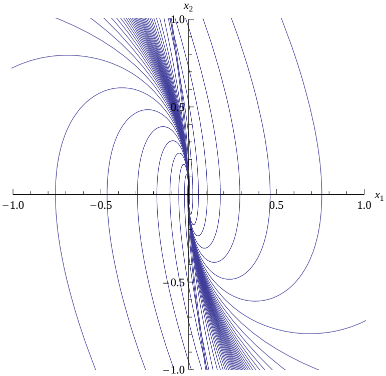
We also have an initial state but we ignore it (for now).

- choose a state $x_T \in \partial G$
- choose a λ_T that satisfies the transversality condition
- solve the state and adjoint equations backward in time
- find unique terminal data such that the trajectory passes though the initial data, $x(0)$

Example - Cont'd

The target set is the origin. However, consider the target set shown below. $\lambda(t_2)^T \delta x(t_2)$





Scale λ so that $\lambda_2^2 = 2$. Backwards in time with $\gamma \to 0$,

$$x_1[t] \to -\sqrt{2}\text{Sign}[\text{Sin}[\theta]](-1 + \text{Cosh}[t] - t\text{Cot}[\theta] + \text{Cot}[\theta]\text{Sinh}[t])$$

$$x_2[t] \to \frac{e^{-t}\left(-1+e^t\right)\left(1+e^t+\left(-1+e^t\right)\text{Cot}[\theta]\right)\text{Sign}[\text{Sin}[\theta]]}{\sqrt{2}}$$

Example: Free Time with Terminal Cost

Consider a system with dynamics

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\end{aligned}$$

and cost function

$$J = x^T(t_f)\,x(t_f) + \int_0^{t_f} \left(1 + u^2\right) dt$$

with free terminal time and free terminal state.

The Hamiltonian is

$$H = 1 + u^2 + \lambda_1 x_2 + \lambda_2 u$$

Example, Cont'd

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^* = -\frac{1}{2}\lambda_2 \Rightarrow H^* = 1 + \lambda_1 x_2 - \frac{\lambda_2^2}{4}$$

from which the adjoint differential equations are obtained:

$$\dot{\lambda}_1 = 0, \dot{\lambda}_2 = -\lambda_1$$

The general solution to the state and adjoint equations are easy to determine:

$$\begin{aligned} &\lambda_1(t) = c_1,\ \ \lambda_2(t) = c_2 - c_1 t \\ &x_1(t) = d_1 + d_2 t - \tfrac{1}{4}c_2 t^2 + \tfrac{1}{12}c_1 t^3, x_2(t) = d_2 - \tfrac{1}{2}c_2 t + \tfrac{1}{4}c_1 t^2 \end{aligned}$$

Direct substitution yields

$$H^* = 1 - \frac{c_2^2}{4} + c_1 d_2$$

which shows that H^* is a constant along optimal trajectories.

Example Cont'd: Steering to the origin

First consider steering from $x_1(0) = 1, x_2(0) = 0$ to $x_1(t_f) = 0, x_2(t_f) = 0$ with free terminal time. With these initial conditions, the state trajectory reduces to:

$$x_1(t) = 1 + \frac{1}{4}c_2 t^2 + \frac{1}{12}c_1 t^3$$
$$x_2(t) = -\frac{1}{2}c_2 t + \frac{1}{4}c_1 t^2$$

and H^* reduces to

$$H^* = 1 - \frac{c_2^2}{4}$$

So, the terminal conditions are

$$1 + \frac{1}{4}c_2 {t_f}^2 + \frac{1}{12}c_1 {t_f}^3 = 0, -\frac{1}{2}c_2 t_f + \frac{1}{4}c_1 {t_f}^2 = 0, 1 - \frac{c_2^2}{4} = 0$$

Solve for $c_1, c_2, t_f \geq 0$

$$c_1 = 2\sqrt{\frac{2}{3}}, c_2 = 2, t_f = \sqrt{6}$$

Example Cont'd: Steering to free terminal state

- As before the same adjoint equations obtain
- As before the free time condition requires $H^* = 0 \Rightarrow c_2 = 2$
- With free terminal state,the two additional terminal conditions are:

$$\lambda\left(t_f\right) = \frac{\partial \ell\left(x\left(t_f\right)\right)}{\partial x} \Rightarrow x_1\left(t_f\right) = \frac{1}{2}\lambda_1\left(t_f\right), x_2\left(t_f\right) = \frac{1}{2}\lambda_2\left(t_f\right)$$