Optimal Control Systems

Constraints

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Outline

Necessary Conditions without Control or State Constraints

Since we require $\delta J = 0$ for arbitrary variations we obtain the key necessary conditions for an optimal trajectory:

$$\begin{aligned} \dot{x}^* &= H_\lambda^T\left(x^*, \lambda^*, u^*\right) && \text{state equations} \\ \dot{\lambda}^{*T} &= -H_x\left(x^*, \lambda^*, u^*\right) && \text{adjoint equations} \\ H_u &= 0 \\ H\left(x^*, \lambda^*, u^*\right) &= c \end{aligned}$$

with the boundary conditions:

- initial state $x\left(t_1\right) = x_0$
- transversality condition $\left(\ell_x - \lambda^T\right)_{t_2} \delta x\left(t_2\right) = 0$
- if the terminal time is free, then $\left[L + \lambda f\right]_{t_2} = H_{t_2} = 0$

Control Constraints

- In this section we address problems with control constraints,

$$u \in U \subset R^m$$

- The important modification to the necessary conditions developed previously is that the condition $H_u = 0$ is replaced by

$$u^* = \arg \min_{u \in U} H\left(x, u, \lambda, t\right)$$

- This result is called the Pontryagin Minimal Principle.
- First, we consider an example

Preliminary Example

Let us reconsider the earlier problem, but with control constraint:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u, \quad |u| \leq 1$$

We wish to steer from an arbitrary initial state to the origin and minimize the cost

$$J = \int_0^{t_2} \left(1 + \tfrac{1}{2}u^2\right) dt$$

As before

$$H = 1 + \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 \left(-x_2 + u\right)$$

Thus,

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1 + \lambda_2$$

But the control is

$$u^* = \arg \min_{|u| \leq 1} \left\{ \frac{1}{2}u^2 + \lambda_2 u \right\}$$

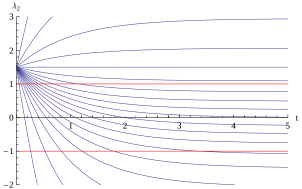
Preliminary Example, Cont'd

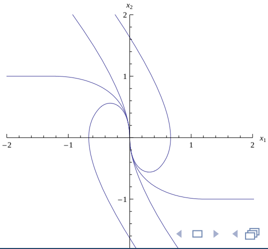
Thus we compute the control:

$$u^* = \begin{cases} -1 & \lambda_2 > 1 \\ -\lambda_2 & -1 \leq \lambda_2 \leq 1 \\ 1 & \lambda_2 < -1 \end{cases} = -\text{sat}\lambda_2$$

We also have the terminal condition

$$H_{t_2} = 1 + \frac{1}{2}\text{sat}^2\lambda_2(t_2) - \lambda_2(t_2)\,\text{sat}\lambda_2(t_2) = 0 \Rightarrow \lambda_2(t_2) = \pm 3/2$$





The Variation with Constraints

- We seek to find a control $u^*(t)$ that minimizes the cost $J(u)$, i.e., $J(u^*) \leq J(u)$ for all admissible u.
- Define $\Delta J \triangleq J(u) - J(u^*)$, then we require $\Delta J \geq 0$.
- Recall from previous calculations for unconstrained control

$$\delta J = \left(\ell_x - \lambda^T\right)_{t_2} \delta x(t_2) + H_{t_2} \delta t_2 + \int_{t_1}^{t_2} \left[\left(H_x + \dot{\lambda}^T\right) \delta x + H_u \delta u + (f - \dot{x})^T \delta \lambda\right] dt$$

  where u, x, λ, t_2 admitted independent and unconstrained variations.
- Now, the variations of x, λ, t_2 remain unconstrained so their coefficients must vanish as before. What needs to be done with δu? First, apply the following replacement

$$H_u(x^*, \lambda^*, u^*)\, \delta u \to H(x^*, \lambda^*, u^* + \delta u) - H(x^*, \lambda^*, u^*)$$

  so that

$$\delta J(u^*) = \int_{t_1}^{t_2} \left[H(x^*, \lambda^*, u^* + \delta u) - H(x^*, \lambda^*, u^*)\right] dt \geq 0$$

The Variation with Constraints, Cont'd

Lemma

It is necessary that

$$H\left(x^*, \lambda^*, u\right) - H\left(x^*, \lambda^*, u^*\right) \geq 0$$

for all admissible $\delta u\left(t\right) = u\left(t\right) - u^*\left(t\right)$, $\|\delta u\| < \varepsilon$ *and all* $t \in [t_1, t_2]$.

The argument goes as follows:

- Let $[t_a, t_b]$ be a nonzero but arbitrarily small subinterval of $[t_1, t_2]$.
- Suppose

$$\begin{aligned} u\left(t\right) &= u^*\left(t\right) && t \notin [t_a, t_b] \\ u\left(t\right) &= u^*\left(t\right) + \delta u\left(t\right) && t \in [t_a, t_b] \end{aligned}$$

  for arbitrary δu, $\|\delta u\| < \varepsilon$.

The Variation with Constraints, Cont'd

- Suppose the desired result is not satisfied on $[t_a, t_b]$, so that

$$H\left(x^*, \lambda^*, u\right) < H\left(x^*, \lambda^*, u^*\right)$$

Then

$$\int_{t_1}^{t_2} \left[H\left(x^*, \lambda^*, u\right) - H\left(x^*, \lambda^*, u^*\right)\right] dt = \int_{t_a}^{t_b} \left[H\left(x^*, \lambda^*, u\right) - H\left(x^*, \lambda^*, u^*\right)\right] dt < 0$$

- Since the interval $[t_a, t_b]$ is arbitrary it follows that if

$$H\left(x^*, \lambda^*, u\right) < H\left(x^*, \lambda^*, u^*\right)$$

for any t, it is possible to construct an admissible variation of u^* such that $\Delta J < 0$, violating the condition for optimality of u^*.

Pontryagin Minimal Principle

Necessary conditions for an optimal control:

- for all $t \in [t_1, t_2]$,

$$\begin{aligned}
\dot{x}^* &= H_\lambda^T \left(x^*, \lambda^*, u^*\right) \\
\dot{\lambda}^T &= -H_x \left(x^*, \lambda^*, u^*\right) \\
H \left(x^*, \lambda^*, u^*\right) &\le H \left(x^*, \lambda^*, u\right), \quad \forall u \in U \\
H \left(x^*, \lambda^*, u^*\right) &= c
\end{aligned}$$

- with boundary conditions:
  - initial state $x(t_1) = x^1$
  - transversality condition $\left(\ell_x - \lambda^T\right)_{t_2} \delta x(t_2) = 0$
  - if the terminal time is free, then $H\left(x^*, \lambda^*, u^*\right)_{t_2} = 0$

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \quad |u| \leq 1 \end{aligned}$$

with cost function

$$J = \int_{t_1}^{t_2} (1 + \beta |u|) \, dt$$

We want to steer the system from an arbitrary initial state to the origin in such a way as to minimize J.

- Note that in spacecraft control problems $\int |u| \, dt$ is referred to as a 'fuel' penalty, whereas $\int u^2 dt$ is a 'power' penalty.
- consequently, the cost J characterizes a tradeoff between time to target and fuel.

Example Cont'd

The Hamiltonian is

$$H = 1 + \beta |u| + \lambda_1 x_2 + \lambda_2 u$$

from which we obtain the adjoint dynamics

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda_1 &= c_1 \\ \lambda_2 &= -c_1 t + c_2 \end{aligned}$$

and

$$u^* = \arg \min_{|u| \le 1} \left[\beta |u| + \lambda_2 u\right]$$

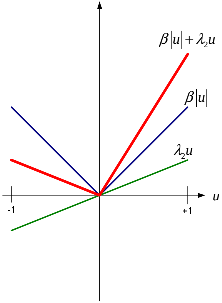
Example, Cont'd

The resulting control law is

$$u^* = \begin{cases} -1 & \lambda_2 > \beta \\ 0 & -\beta < \lambda_2 < \beta \\ 1 & \lambda_2 < -\beta \end{cases}$$

- Since λ_2 is a linear function of t, there can be at most 2 switches.

A free terminal time requires



$$H(t_2) = 1 + \beta\,|u(t_2)| + \lambda_2 u(t_2) = 0$$

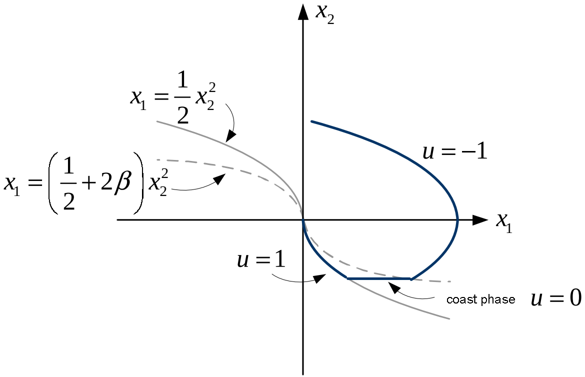
Thus, it is not possible to have $u^*(t_2) = 0$. Furthermore, $u^*(t_2) = 1$ implies

$$1 + \beta + \lambda_2 = 0 \Rightarrow \lambda_2(t_2) = -(1+\beta) < \beta$$

and $u^*(t_2) = -1$ implies

$$1 + \beta - \lambda_2 = 0 \Rightarrow \lambda_2(t_2) = (1+\beta) > \beta$$

Example - Fixed Time, Control Constraints



State Constraints

- Here, we consider problems in which the state is restricted to a subset of $S \subset R^n$.
- It will be assumed that the the allowable domain can be defined by a set of inequalities

$$\phi_1(x,t) \geq 0, \ldots, \phi_s(x,t) \geq 0$$

where each ϕ_i is a smooth function of x, t.
- Introduce a new state variable $x_{n+1}(t)$, defined by

$$\dot{x}_{n+1}(t) = \phi_1^2(x,t)\, u_0(-\phi_1) + \cdots + \phi_s^2(x,t)\, u_0(-\phi_s)\,, \quad u_0 \text{ denotes the unit step}$$

with boundary conditions

$$x_{n+1}(t_1) = 0, x_{n+1}(t_2) = 0$$

note that these boundary conditions can be satisfied only if the constraints are satisfied along the entire trajectory.
- The necessary condition stated above can be applied, with the additional state equation.

Satisfying the Necessary Conditions

Given an initial state there are two possibilities:

- There does not exist any trajectory that satisfies all of the necessary conditions including the state constraints
- There exists one or more optimal trajectories, and these can be of two types
  - the entire trajectory lies interior to the state constraint set S
  - trajectory segments of finite length lie on the boundary of S

Example

Once again consider the problem:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u, \quad |u| \leq 1\end{aligned}$$

We wish to steer from an arbitrary initial state to the origin and minimize the cost

$$J = \int_0^{t_2} \left(1 + \tfrac{1}{2}u^2\right) dt$$

However, in this case we impose the state constraint

$$-2 \leq x_2 \leq 2 \Leftrightarrow (x_2 + 2)(2 - x_2) \geq 0$$

Example, Cont'd

The expanded dynamics are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$
$$\dot{x}_3 = (x_2 + 2)^2 (2 - x_2)^2 u_0 ((x_2 + 2)(x_2 - 2))$$

From which we obtain the Hamiltonian:

$$H = 1 + \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 (-x_2 + u) + \lambda_3 (x_2 + 2)^2 (2 - x_2)^2 u_0 ((x_2 + 2)(x_2 - 2))$$

Also as before (see preliminary example) the optimal control is

$$u^* = -\mathrm{sat}\lambda_2$$

Example, Cont'd

The adjoint equations are derived from H

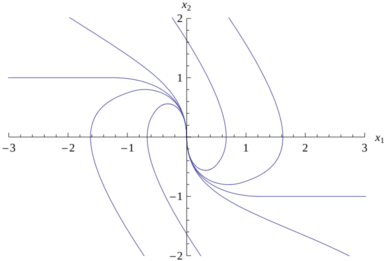
$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= \begin{cases} -\lambda_1 + \lambda_2 & -2 < x_2 < 2 \\ -\lambda_1 + \lambda_2 - 4x_2\left(x_2^2 - 4\right)\lambda_3 & x_2 < -2 \vee x_2 > 2 \end{cases} \\ \dot{\lambda}_3 &= 0 \end{aligned}$$

And, we have the unchanged terminal condition

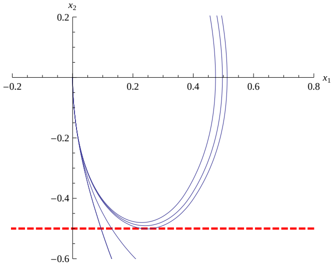
$$H_{t_2} = 1 + \frac{1}{2}\mathrm{sat}^2\lambda_2\left(t_2\right) - \lambda_2\left(t_2\right)\mathrm{sat}\lambda_2\left(t_2\right) = 0 \Rightarrow \lambda_2\left(t_2\right) = \pm 3/2$$

Example Cont'd

Trajectories computed via *backing out of target*. Clearly, the state constraint, $-2 < x_2 < 2$ has no effect.



With the state constraint changed to $-0.5 < x_2 < 2$, the trajectories are clearly altered.



Motion on the Constraint Boundary

Now, we will consider optimal trajectory segments on the constraint boundary. Suppose the allowable region $S \subset R^n$ is characterized by the scalar-valued function $\phi(x)$:

$$S = \{x \in R^n \,|\phi(x) \leq 0\}$$

Define

$$c(x, u) = \frac{\partial \phi(x)}{\partial x} f(x, u)$$

If $x(t) \in \partial S$ for some finite time interval $t_a \leq t \leq t_b$ then $c(x, u) \equiv 0$ on $t_a \leq t \leq t_b$. This simply means that $\dot{x}$ lies in the tangent plane to the surface ∂S.

Optimal motion on the Constraint Boundary

Lemma (Minimal Principle on Boundary)

Suppose $u^*(t)$ *and* $x^*(t)$ *are optimal and* $x^*(t) \in \partial S$ *for* $t_a \leq t \leq t_b$*. Then there exists adjoint variables* $\lambda(t)$ *and a scalar valued function* $\alpha(t)$ *such that on* $t \in [t_a, t_b]$

$$\dot{\lambda}^* = -\frac{\partial H(x^*, \lambda^*, u^*)}{\partial x} - \alpha \frac{\partial c(x^*, u^*)}{\partial x}$$

$$H(x^*, \lambda^*, u^*) = \min_{u \in U} \{H(x^*, \lambda^*, u) \,|\, c(x, u) = 0\}$$

$$u^*, \alpha^* = \arg \min_{u \in U, \alpha \in R} \{H(x^*, \lambda^*, u) + \alpha c(x^*, u)\}$$

Remarks

- If $x^*(t) \in \text{int}S$ for times $t_1 \leq t < t_a$, then the ordinary minimum principle holds therein.
- The adjoint variables are continuous at t_a, i.e.,

$$\lambda\left(t_a^-\right) = \lambda\left(t_a^+\right)$$

- However, when leaving the boundary

$$\lambda\left(t_b^+\right) = \lambda\left(t_b^-\right) - \alpha\left(t_b\right) \frac{\partial \phi\left(x\left(t_b\right)\right)}{\partial x}$$

Example

Consider the problem of motion in a plane defined dynamics

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$

with control constraint

$$U = \left\{ u \in R^2 \,\middle|\, u_1^2 + u_2^2 - 1 \leq 0 \right\}$$

We seek to steer an arbitrary state to the origin in minimum time

$$J = \int_{t_1}^{t_2} dt$$

while avoiding the obstacle - a unit circle centered at $(2, 0)$. The admissible space is defined by

$$\phi\left(x, t\right) = 1 - \left(x_1 - 2\right)^2 - x_2^2 \leq 0$$

Example, Cont'd

The ordinary minimum principle which holds while the trajectory is inside the admissible space, S, i.e., outside of the circle.

$$H(x, \lambda, u) = 1 + \lambda_1 u_1 + \lambda_2 u_2$$

so that

$$u_1^* = -\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad u_2^* = -\frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

and the adjoint equations are

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = 0$$

and we require the free time condition

$$H(t_2) = \lambda_1^2(t_2) + \lambda_2^2(t_2) - 1 = 0 \Rightarrow \lambda_1(t_2), \lambda_2(t_2) \text{ on unit circle}$$

Example Cont'd

Without the obstacle we have

$$\lambda_1(t) = \cos\theta, \quad \lambda_2(t) = \sin\theta$$
$$u_1^* = -\cos\theta, \quad u_2^* = -\sin\theta$$

All trajectories are straight lines!

Now suppose that $x^*(t) \in \partial S$ for $t_a \leq t \leq t_b$. Compute

$$c(x,u) = -2(x_1 - 2)u_1 - 2x_2u_2 = 0$$

and

$$\dot{\lambda}_1 = 2\alpha u_1, \quad \dot{\lambda}_2 = 2\alpha u_2$$

Example, Cont'd

Now, we need to find u^* by minimizing H subject to the constraint $c(x,u)=0$.

$$\min_{u\in U,\alpha\in R}\{H(x,\lambda,u)+\alpha c(x,u)\}$$

First, consider a transformation

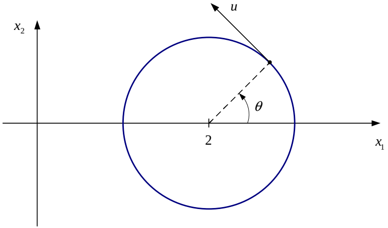
$$x_1 \to 2+\cos\theta,\ x_2 \to \sin\theta$$

so that points on ∂S are parameterized by θ as shown in the figure.

There are two solutions

$$\begin{aligned} u_1 &= -\sin\theta,\ u_2 = \cos\theta, \\ \alpha &= (\lambda_1\cos\theta+\lambda_2\sin\theta)/2 \\ u_1 &= \sin\theta,\ u_2 = -\cos\theta, \\ \alpha &= (\lambda_1\cos\theta+\lambda_2\sin\theta)/2 \end{aligned}$$

The first minimizes H on the upper half circle and the second on the lower half circle.



Example: Obstacle Approach

Let us focus on the top half circle. Consider the two relations

$$H^* = 1 + \lambda_1 u_1 + \lambda_2 u_2 \equiv 0, \quad \alpha = (\lambda_1 u_2 - \lambda_2 u_1)/2$$

Use these equations and the first of the adjoint differential equations

$$\dot{\lambda}_1 = 2\alpha u_1$$

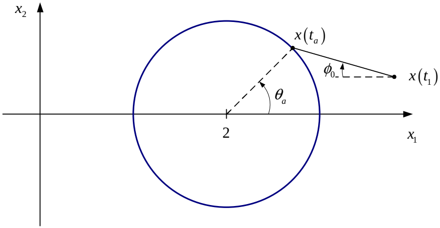
to obtain

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_1 \tan\theta + \sin\theta \tan\theta \\ \lambda_2 &= \lambda_1 \tan\theta - \sec\theta \end{aligned}$$

Now, solve the differential equation first to obtain

$$\begin{aligned} \lambda_1 &= c_1 \cos\theta - \theta\cos\theta + \sin\theta \\ \lambda_2 &= c_1 \sin\theta - \theta\sin\theta - \cos\theta \end{aligned}$$

Example: Obstacle Approach, Cont'd

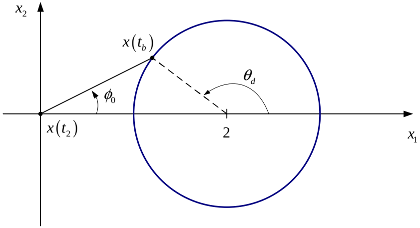


Continuity of the adjoint variables yields

$$\lambda_1\left(t_a^-\right) = \lambda_1\left(t_a^+\right) \Rightarrow \cos\phi_0 \;= c_1\cos\theta_a - \theta_a\cos\theta_a + \sin\theta_a$$
$$\lambda_2\left(t_a^-\right) = \lambda_2\left(t_a^+\right) \Rightarrow -\sin\phi_0 \;= c_1\sin\theta_a - \theta_a\sin\theta_a - \cos\theta_a$$

$$c_1 = \theta_a, \quad \theta_a + \phi_0 = \frac{\pi}{2}$$

Example: Obstacle Departure



Just before departure we have

$$\lambda_1\left(t_b^-\right) = \theta_a \cos\theta_d - \theta_d \cos\theta_d + \sin\theta_d$$
$$\lambda_2\left(t_b^-\right) = \theta_a \sin\theta_d - \theta_d \sin\theta_d - \cos\theta_d$$

Just after departure, since the usual equations obtain, we have

$$\lambda_1\left(t_b^+\right) = \cos\phi_0, \quad \lambda_2\left(t_b^+\right) = \sin\phi_0$$

Example: Obstacle Departure, Cont'd

Now, we need to satisfy the departure continuity equations

$$\lambda\left(t_b^+\right) = \lambda\left(t_b^-\right) - \alpha\left(t_b\right)\frac{\partial\varphi\left(x\left(t_b\right)\right)}{\partial x}$$

Compute:

$$\left.\frac{\partial\varphi}{\partial x}\right|_{\theta_d} = \begin{bmatrix} -2\left(x_1 - 2\right) & -2x_2 \end{bmatrix} = \begin{bmatrix} -2\cos\theta_d & -2\sin\theta_d \end{bmatrix}$$

$$\alpha|_{\theta_d} = \frac{1}{2}\left(\lambda_1\left(t_d^-\right)\cos\theta_d + \lambda_2\left(t_d^-\right)\sin\theta_d\right) = \frac{\theta_a - \theta_d}{2}$$

Consequently,

$$\begin{aligned} \lambda_1\left(t_b^+\right) = \lambda_1\left(t_b^-\right) - \alpha\left(t_b\right)\tfrac{\partial\varphi(x(t_b))}{\partial x_1} \Rightarrow \cos\phi_0 &= \sin\theta_d \\ \lambda_2\left(t_b^+\right) = \lambda_2\left(t_b^-\right) - \alpha\left(t_b\right)\tfrac{\partial\varphi(x(t_b))}{\partial x_2} \Rightarrow \sin\phi_0 &= -\cos\theta_d \end{aligned}$$

$$\Rightarrow \phi_0 = \pi/6, \quad \theta_d = 2\pi/3$$

Example: Optimal Trajectories