Optimal Control Systems

Examples

Harry G. Kwatny

Department of Mechanical Engineering & Mechanics
Drexel University

Outline

Pontryagin Minimal Principle

Necessary conditions for an optimal control:

- for all $t \in [t_1, t_2]$,

$$\begin{aligned}
\dot{x}^* &= H_\lambda^T(x^*, \lambda^*, u^*) \\
\dot{\lambda}^T &= -H_x(x^*, \lambda^*, u^*) \\
H(x^*, \lambda^*, u^*) &\le H(x^*, \lambda^*, u), \quad \forall u \in U \\
H(x^*, \lambda^*, u^*) &= c
\end{aligned}$$

- with boundary conditions:
  - initial state $x(t_1) = x^1$
  - transversality condition $\left(\ell_x - \lambda^T\right)_{t_2} \delta x(t_2) = 0$
  - if the terminal time is free, then $H(x^*, \lambda^*, u^*)_{t_2} = 0$

PRELIMINARY EXAMPLE

Let us reconsider the earlier problem, but with control constraint:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u, \quad |u| \leq 1\end{aligned}$$

We wish to steer from an arbitrary initial state to the origin and minimize the cost

$$J = \int_0^{t_2} \left(1 + \tfrac{1}{2}u^2\right) dt$$

As before

$$H = 1 + \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 \left(-x_2 + u\right)$$

Thus,

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1 + \lambda_2$$

But the control is

$$u^* = \arg \min_{|u| \leq 1} \left\{ \frac{1}{2}u^2 + \lambda_2 u \right\}$$

Preliminary Example, Cont'd

Thus we compute the control:

$$u^* = \begin{cases} -1 & \lambda_2 > 1 \\ -\lambda_2 & -1 \le \lambda_2 \le 1 \\ 1 & \lambda_2 < -1 \end{cases} = -\text{sat}\lambda_2$$

We also have the terminal condition

$$H_{t_2} = 1 + \frac{1}{2}\text{sat}^2\lambda_2(t_2) - \lambda_2(t_2)\,\text{sat}\lambda_2(t_2) = 0 \Rightarrow \lambda_2(t_2) = \pm 3/2$$

```
Eq = 1 + Sat[μ]^2 / 2 - μ Sat[μ] == 0;
Reduce[Eq, μ, Reals]
```

$\mu == -\frac{3}{2} \;||\; \mu == \frac{3}{2}$

```
solλ = DSolve[{D[λ1[t], t] == 0, D[λ2[t], t] == λ1[t] - λ2[t], λ2[0] == Sin[θ] (3 / 2) / Abs[Sin[θ]],
   λ1[0] == Cos[θ] (3 / 2) / Abs[Sin[θ]]}, {λ1[t], λ2[t]}, t]
```

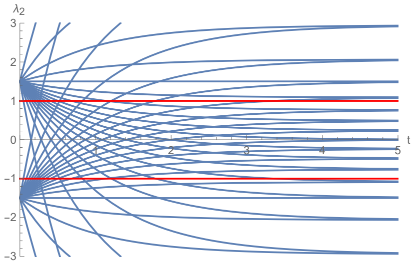
$\left\{\left\{\lambda1[t] \to \frac{3\,\text{Cos}[\theta]}{2\,\text{Abs}[\text{Sin}[\theta]]},\ \lambda2[t] \to \frac{3\,e^{-t}\left(-\text{Cos}[\theta] + e^{t}\,\text{Cos}[\theta] + \text{Sin}[\theta]\right)}{2\,\text{Abs}[\text{Sin}[\theta]]}\right\}\right\}$

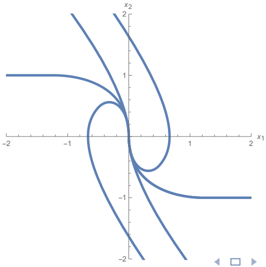
Preliminary Example, Cont'd

$$H_{t_2} = 1 + \frac{1}{2}\text{sat}^2\lambda_2(t_2) - \lambda_2(t_2)\,\text{sat}\lambda_2(t_2) = 0 \Rightarrow \lambda_2(t_2) = \pm 3/2$$

$$\left(\ell_x - \lambda^T\right)_{t_2}\delta x(t_2) = 0 \Rightarrow \lambda^T(t_2)\left.\frac{\partial\phi(x)}{\partial x}\right|_{x=x^*},\ \phi(x) = x_1^2 + x_2^2 - \varepsilon^2 = 0$$

```
PlotList = Join[Range[π / 20, π (1 - 1 / 20), π / 20], Range[-π / 20, -π (1 - 1 / 20), -π / 20]];
gr1 = Plot[Map[(({λ2[t]} /. solλ) /. θ → #) &, PlotList], {t, 0, 5}, PlotRange → {{0, 5}, {-3, 3}},
   AxesLabel → {"t", "λ₂"}];
gr2 = Plot[{1, -1}, {t, 0, 5}, PlotStyle → {{Red}}];
Show[gr1, gr2]
```





State Constraints

- Here, we consider problems in which the state is restricted to a subset of $S \subset R^n$.
- It will be assumed that the the allowable domain can be defined by a set of inequalities

$$\phi_1(x,t) \geq 0, \ldots, \phi_s(x,t) \geq 0$$

where each ϕ_i is a smooth function of x, t.
- Introduce a new state variable $x_{n+1}(t)$, defined by

$$\dot{x}_{n+1}(t) = \phi_1^2(x,t)\, u_0(-\phi_1) + \cdots + \phi_s^2(x,t)\, u_0(-\phi_s)\,, \quad u_0 \text{ denotes the unit step}$$

with boundary conditions

$$x_{n+1}(t_1) = 0, x_{n+1}(t_2) = 0$$

note that these boundary conditions can be satisfied only if the constraints are satisfied along the entire trajectory.
- The necessary condition stated above can be applied, with the additional state equation.

Satisfying the Necessary Conditions

Given an initial state there are two possibilities:

- There does not exist any trajectory that satisfies all of the necessary conditions including the state constraints
- There exists one or more optimal trajectories, and these can be of two types
  - the entire trajectory lies interior to the state constraint set S
  - trajectory segments of finite length lie on the boundary of S

MOTION ON THE CONSTRAINT BOUNDARY

Now, we will consider optimal trajectory segments on the constraint boundary. Suppose the allowable region $S \subset R^n$ is characterized by the scalar-valued function $\phi(x)$:

$$S = \{x \in R^n \,|\phi(x) \leq 0\}$$

Define

$$c(x, u) = \frac{\partial \phi(x)}{\partial x} f(x, u)$$

If $x(t) \in \partial S$ for some finite time interval $t_a \leq t \leq t_b$ then $c(x, u) \equiv 0$ on $t_a \leq t \leq t_b$.
This simply means that $\dot{x}$ lies in the tangent plane to the surface ∂S.

Optimal motion on the Constraint Boundary

Lemma (Minimal Principle on Boundary)

Suppose $u^*(t)$ *and* $x^*(t)$ *are optimal and* $x^*(t) \in \partial S$ *for* $t_a \le t \le t_b$*. Then there exists adjoint variables* $\lambda(t)$ *and a scalar valued function* $\alpha(t)$ *such that on* $t \in [t_a, t_b]$

$$\dot{\lambda}^* = -\frac{\partial H(x^*, \lambda^*, u^*)}{\partial x} - \alpha \frac{\partial c(x^*, u^*)}{\partial x}$$

$$H(x^*, \lambda^*, u^*) = \min_{u \in U} \{H(x^*, \lambda^*, u) \mid c(x, u) = 0\}$$

$$u^*, \alpha^* = \arg \min_{u \in U, \alpha \in R} \{H(x^*, \lambda^*, u) + \alpha c(x^*, u)\}$$

REMARKS

- If $x^*(t) \in \text{int}S$ for times $t_1 \leq t < t_a$, then the ordinary minimum principle holds therein.
- The adjoint variables are continuous at t_a, i.e.,

$$\lambda\left(t_a^-\right) = \lambda\left(t_a^+\right)$$

- However, when leaving the boundary

$$\lambda\left(t_b^+\right) = \lambda\left(t_b^-\right) - \alpha\left(t_b\right)\frac{\partial\phi\left(x\left(t_b\right)\right)}{\partial x}$$

EXAMPLE

Consider the problem of motion in a plane defined dynamics

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$

with control constraint

$$U = \left\{ u \in R^2 \left| u_1^2 + u_2^2 - 1 \leq 0 \right. \right\}$$

We seek to steer an arbitrary state to the origin in minimum time

$$J = \int_{t_1}^{t_2} dt$$

while avoiding the obstacle - a unit circle centered at $(2, 0)$. The admissible space is defined by

$$\phi(x, t) = 1 - (x_1 - 2)^2 - x_2^2 \leq 0$$

Example, Cont'd

The ordinary minimum principle which holds while the trajectory is in the interior of S:

$$H(x, \lambda, u) = 1 + \lambda_1 u_1 + \lambda_2 u_2$$

so that

$$u_1^* = -\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad u_2^* = -\frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}$$

and the adjoint equations are

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = 0$$

and we require the free time condition

$$H(t_2) = \lambda_1^2(t_2) + \lambda_2^2(t_2) - 1 = 0 \Rightarrow \lambda_1(t_2), \lambda_2(t_2) \text{ are on a unit circle in } \lambda - \text{space}$$

Example Cont'd

Without the obstacle we have

$$\lambda_1(t) = \cos\theta, \quad \lambda_2(t) = \sin\theta$$
$$u_1^* = -\cos\theta, \quad u_2^* = -\sin\theta$$

All trajectories are straight lines!

Now suppose that $x^*(t) \in \partial S$ for $t_a \leq t \leq t_b$. Compute

$$c(x, u) = -2(x_1 - 2)u_1 - 2x_2u_2 = 0$$

and

$$\dot{\lambda}_1 = 2\alpha u_1, \quad \dot{\lambda}_2 = 2\alpha u_2$$

EXAMPLE, CONT'D

Now, we need to find u^* by minimizing H subject to the constraint $c(x, u) = 0$.

$$\min_{u \in U, \alpha \in R} \{H(x, \lambda, u) + \alpha c(x, u)\}$$

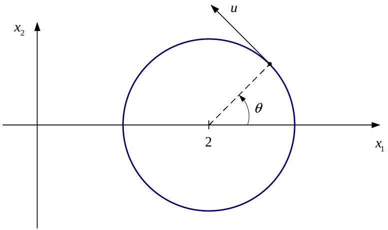
First, consider a transformation

$$x_1 \to 2 + \cos\theta, \; x_2 \to \sin\theta$$

so that points on ∂S are parameterized by θ as shown in the figure.

There are two solutions

$$\begin{aligned} u_1 &= -\sin\theta, \; u_2 = \cos\theta, \\ \alpha &= (\lambda_1 \cos\theta + \lambda_2 \sin\theta)/2 \\ u_1 &= \sin\theta, \; u_2 = -\cos\theta, \\ \alpha &= (\lambda_1 \cos\theta + \lambda_2 \sin\theta)/2 \end{aligned}$$



The first minimizes H on the upper half circle and the second on the lower half circle.

Example: Obstacle Approach

Let us focus on the top half circle. Consider the two relations

$$H^* = 1 + \lambda_1 u_1 + \lambda_2 u_2 \equiv 0, \quad \alpha = (\lambda_1 u_2 - \lambda_2 u_1)/2$$

Use these equations and the first of the adjoint differential equations

$$\dot{\lambda}_1 = 2\alpha u_1$$

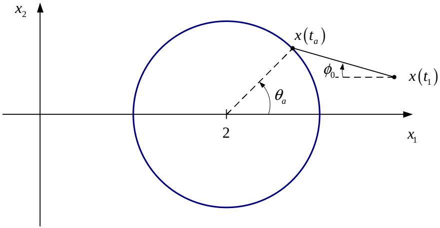
to obtain

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_1 \tan\theta + \sin\theta \tan\theta \\ \lambda_2 &= \lambda_1 \tan\theta - \sec\theta \end{aligned}$$

Now, solve the differential equation first to obtain

$$\begin{aligned} \lambda_1 &= c_1 \cos\theta - \theta\cos\theta + \sin\theta \\ \lambda_2 &= c_1 \sin\theta - \theta\sin\theta - \cos\theta \end{aligned}$$

Example: Obstacle Approach, Cont'd

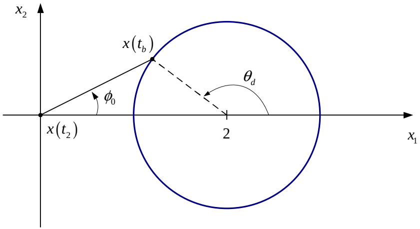


Continuity of the adjoint variables yields

$$\lambda_1\left(t_a^-\right) = \lambda_1\left(t_a^+\right) \Rightarrow \cos\phi_0 = c_1\cos\theta_a - \theta_a\cos\theta_a + \sin\theta_a$$
$$\lambda_2\left(t_a^-\right) = \lambda_2\left(t_a^+\right) \Rightarrow -\sin\phi_0 = c_1\sin\theta_a - \theta_a\sin\theta_a - \cos\theta_a$$

$$c_1 = \theta_a, \quad \theta_a + \phi_0 = \frac{\pi}{2}$$

Example: Obstacle Departure



Just before departure we have

$$\lambda_1\left(t_b^-\right) = \theta_a \cos\theta_d - \theta_d \cos\theta_d + \sin\theta_d$$
$$\lambda_2\left(t_b^-\right) = \theta_a \sin\theta_d - \theta_d \sin\theta_d - \cos\theta_d$$

Just after departure, since the usual equations obtain, we have

$$\lambda_1\left(t_b^+\right) = \cos\phi_0, \quad \lambda_2\left(t_b^+\right) = \sin\phi_0$$

Example: Obstacle Departure, Cont'd

Now, we need to satisfy the departure continuity equations

$$\lambda\left(t_b^+\right) = \lambda\left(t_b^-\right) - \alpha\left(t_b\right) \frac{\partial\varphi\left(x\left(t_b\right)\right)}{\partial x}$$

Compute:

$$\left.\frac{\partial\varphi}{\partial x}\right|_{\theta_d} = \begin{bmatrix} -2\left(x_1 - 2\right) & -2x_2 \end{bmatrix} = \begin{bmatrix} -2\cos\theta_d & -2\sin\theta_d \end{bmatrix}$$

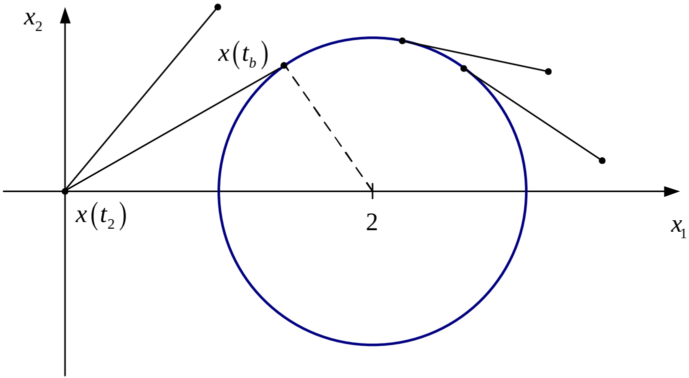
$$\alpha|_{\theta_d} = \frac{1}{2}\left(\lambda_1\left(t_d^-\right)\cos\theta_d + \lambda_2\left(t_d^-\right)\sin\theta_d\right) = \frac{\theta_a - \theta_d}{2}$$

Consequently,

$$\begin{aligned} \lambda_1\left(t_b^+\right) = \lambda_1\left(t_b^-\right) - \alpha\left(t_b\right)\tfrac{\partial\varphi(x(t_b))}{\partial x_1} &\Rightarrow \cos\phi_0 = \sin\theta_d \\ \lambda_2\left(t_b^+\right) = \lambda_2\left(t_b^-\right) - \alpha\left(t_b\right)\tfrac{\partial\varphi(x(t_b))}{\partial x_2} &\Rightarrow \sin\phi_0 = -\cos\theta_d \end{aligned}$$

$$\Rightarrow \phi_0 = \pi/6, \quad \theta_d = 2\pi/3$$

Example: Optimal Trajectories



Example – Moon Lander

Consider the decent of a moon lander.

$$\dot{h} = v$$
$$\dot{v} = -g + k\frac{u}{m}$$
$$\dot{m} = -u$$

The thrust u is used to steer the system to $h = 0, v = 0$. In addition we wish to minimize the fuel used during landing, i.e.

$$J = \int_0^t |u| \, dt$$

or, minimize time,

$$J = \int_0^t dt$$

Control constraint: $0 \leq u \leq c$, *state constraints:* $h \geq 0$, $m > 0$

Moon Lander, Necessary Conditions, min time

$$H\left(x,u,\lambda\right) = 1 + \lambda_1 x_2 + \lambda_2 \left(-g + \frac{k}{x_3} u\right) + \lambda_3 u$$

From which,

$$\begin{aligned} u^* &= \arg\min_u \left[\left(\lambda_2 + \lambda_3 \tfrac{x_3}{k}\right) u\right] = c u_0 \left(-\left(\lambda_2 \tfrac{k}{x_3} + \lambda_3\right)\right) \\ H^* &= \lambda_1 x_2 - \lambda_2 g + \left(\lambda_2 + \lambda_3 \tfrac{x_3}{k}\right) c u_0 \left(-\left(\lambda_2 + \lambda_3 \tfrac{x_3}{k}\right)\right) \end{aligned}$$

and

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= \lambda_1 \\ \dot{\lambda}_3 &= -\lambda_2 \tfrac{k}{x_3^2} c u_0 \left(-\left(\lambda_2 + \lambda_3 \tfrac{x_3}{k}\right)\right) \end{aligned}$$

Moon Lander, Necessary Conditions, min fuel

$$H\left(x,u,\lambda\right) = |u| + \lambda_1 x_2 + \lambda_2\left(-g + \frac{k}{m}x_3\right) + \lambda_3 u$$

From which,

$$\begin{aligned} u^* &= \arg\min_u \left[|u| + \left(\lambda_2 \tfrac{k}{x_3} + \lambda_3\right) u\right] = c u_0 \left(-\left(1 + \lambda_2 \tfrac{k}{x_3} + \lambda_3\right)\right) \\ H^* &= \lambda_1 x_2 - \lambda_2 g + \left(1 + \lambda_2 \tfrac{k}{x_3} + \lambda_3\right) c u_0 \left(-\left(1 + \lambda_2 \tfrac{k}{x_3} + \lambda_3\right)\right) \end{aligned}$$

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= \lambda_1 \\ \dot{\lambda}_3 &= -\lambda_2 \tfrac{k}{x_3^2} c u_0 \left(-\left(1 + \lambda_2 \tfrac{k}{x_3} + \lambda_3\right)\right) \end{aligned}$$