OPTIMAL CONTROL SYSTEMS

MIN-MAX OPTIMAL CONTROL

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Outline

PROBLEM SETUP

Consider the control system

$$\dot{x} = f(x, u, w)$$

where

- $x \in R^n$ is the state
- $u \in U \subset R^m$ is the control
- $w \in W \subset R^q$ is an external disturbance

We seek a control u that minimizes the cost J for the worst possible disturbance w and

$$J = \ell(x(T)) + \int_0^T L(x(t), u(t), w(t))\, dt$$

HJB EQUATION

We seek both $u^*(t)$ and $w^*(t)$ such that

$$\{u^*, w^*\} = \arg \min_{u \subset U} \max_{w \subset W} J(u, w)$$

Define the cost to go

$$V(x,t) = \min_{u \subset U} \max_{w \subset W} \left(\ell(x(T)) + \int_t^T L(x(\tau), u(\tau), w(\tau))\, d\tau \right)$$

The principle of optimality leads to

$$\frac{\partial V(x,t)}{\partial t} + H^*\left(x, \frac{\partial V(x,t)}{\partial x}\right) = 0$$

where

$$H^*(x, \lambda) = \min_{u \in U} \max_{w \in W} \left\{ L(x,u,w) + \lambda^T f(x,u,w) \right\}$$

EXAMPLE

Consider the linear dynamics

$$\dot{x} = Ax + Bu + Ew$$
$$y = Cx$$

and cost

$$J(u, w) = \lim_{T \to \infty} \frac{1}{2T} \int_0^T \left[y^T(t)y(t) + \rho u^T(t)u(t) - \gamma^2 w^T(t)w(t) \right] dt$$

Suppose

- (A, B) and (A, E) are stabilizable
- (A, C) is detectable

Then

$$u(t) = -\frac{1}{\rho} B^T S x(t), \quad w(t) = \frac{1}{\gamma^2} E^T S x(t)$$

$$A^T S + SA - S \left(\frac{1}{\rho} BB^T - \frac{1}{\gamma^2} EE^T \right) S = -C^T C$$

EXAMPLE CONT'D: SOME CALCULATIONS

$$H^* = \min_u \max_w \left\{ \frac{1}{2} \left(x^T C^T C x + \rho u^T u - \gamma^2 w^T w \right) + \lambda^T \left(Ax + Bu + Ew \right) \right\}$$

$$u^* = -\frac{1}{\rho} B^T \lambda, \quad w^* = \frac{1}{\gamma^2} E^T \lambda$$

$$H^* = \frac{1}{2} x^T C^T C x + \lambda^T A x - \frac{1}{2\rho} \lambda^T B B^T \lambda + \frac{1}{2\gamma^2} \lambda^T E E^T \lambda$$

$$\dot{x} = \frac{\partial H^*}{\partial \lambda} = Ax + \frac{1}{\gamma^2} E E^T \lambda - \frac{1}{\rho} B B^T \lambda$$

$$\dot{\lambda} = -\frac{\partial H^*}{\partial x} = -C^T C x + -A^T \lambda$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & \frac{1}{\gamma^2} E E^T - \frac{1}{\rho} B B^T \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

EXAMPLE CONT'D

Consider the Hamiltonian matrix

$$H = \begin{bmatrix} A & \frac{1}{\gamma^2}EE^T - \frac{1}{\rho}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

Now, under the above assumptions:

- There exists a γ_{min} such that so that there are no eigenvalues of H on the imaginary axis provided $\gamma > \gamma_{\min}$.
- When $\gamma \to \infty$ we approach the standard LQR solution.
- For $\gamma = \gamma_{\min}$, the controller is the full state feedback H_∞ controller.
- All other values of γ produce valid min-max controllers.
- The condition $\text{Re}\lambda(H) \neq 0$ is equivalent to stability of the matrix

$$A + \frac{1}{\gamma^2}EE^T - \frac{1}{\rho}BB^T$$

- The closed loop system matrix is

$$A - \frac{1}{\rho}BB^T$$

The term EE^T/γ^2 is destabilizing, so the system is guaranteed some margin of stability.

Two Person Differential Game

Once again consider the system

$$\dot{x} = f\left(x, u, w\right)$$

However, now consider both u and w to be controls associated, respectively, with two players A and B.
Player A wants to minimize the cost

$$J = \ell\left(x\left(T\right)\right) + \int_0^T L\left(x\left(t\right), u\left(t\right), w\left(t\right)\right) dt$$

and player B wants to maximize it. This is a two person, zero-sum, differential game.

DEFINITIONS

Instead of the 'cost to go' define two value functions:

DEFINITION (VALUE FUNCTIONS)

The lower value function is

$$V_A\left(x,t\right)=\min_{u\subset U}\max_{w\subset W}\left(\ell\left(x\left(T\right)\right)+\int_t^T L\left(x\left(\tau\right),u\left(\tau\right),w\left(\tau\right)\right)d\tau\right)$$

and the upper value function is

$$V_B\left(x,t\right)=\max_{w\subset W}\min_{u\subset U}\left(\ell\left(x\left(T\right)\right)+\int_t^T L\left(x\left(\tau\right),u\left(\tau\right),w\left(\tau\right)\right)d\tau\right)$$

- if $V_A=V_B$, a unique solution exists. This solution is called a pure strategy.
- If $V_A\neq V_B$, a unique strategy does not exist. In general, the player who 'plays second' has an advantage.

Isaacs' Equations

Define

$$H_A(x,\lambda) = \min_{u\in U}\max_{w\in W}\left\{L(x,u,w) + \lambda^T f(x,u,w)\right\}$$

$$H_B(x,\lambda) = \min_{u\in U}\max_{w\in W}\left\{L(x,u,w) + \lambda^T f(x,u,w)\right\}$$

Then we get Isaacs' equations

$$\frac{\partial V_A(x,t)}{\partial t} + H_A\left(x, \frac{\partial V_A(x,t)}{\partial x}\right) = 0$$

$$\frac{\partial V_B(x,t)}{\partial t} + H_B\left(x, \frac{\partial V_B(x,t)}{\partial x}\right) = 0$$

$$H_A\left(x, \frac{\partial V_A(x,t)}{\partial x}\right) \equiv H_B\left(x, \frac{\partial V_B(x,t)}{\partial x}\right) \Rightarrow V_A(x,t) \equiv V_B(x,t)$$

and the game is said to satisfy the Isaacs' condition. Notice that if u^*,w^* are optimal,the the pair (u^*, w^*) is a saddle point in the sense that

$$J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*)$$

Well-stirred Heating Tank Startup

$$\dot{x} = -x + w + u, \quad x(0) = x_0, \quad 0 \le u \le 2, \quad |w| \le 1$$

- x, dimensionless tank temperature
- u, dimensionless heat input
- w, dimensionless input water flow

$$J = \int_0^T (x - \bar{x})^2 dt, \quad T \text{ fixed}$$

$$H = (x - \bar{x})^2 + \lambda(-x + w + u)$$

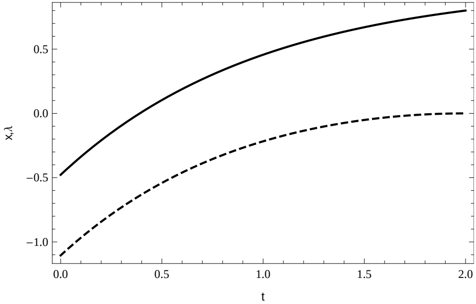
$$\{u^*, w^*\} = \arg \min_u \max_w H(x, u, w, \lambda)$$

$$\Rightarrow u^* = (1 - \text{sgn}(\lambda)), w^* = \text{sgn}(\lambda)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \Rightarrow \dot{\lambda} = \lambda - 2(x - \bar{x}), \quad \lambda(T) = 0$$

EXAMPLE, CONT'D

$$\dot{x} = -x + 1$$
$$\dot{\lambda} = \lambda - 2\left(x - \bar{x}\right)$$
$$x\left(T\right) = \bar{x}, \lambda\left(T\right) = 0$$



with $\bar{x} = 0.8$