Optimal Control Systems

Hybrid Systems

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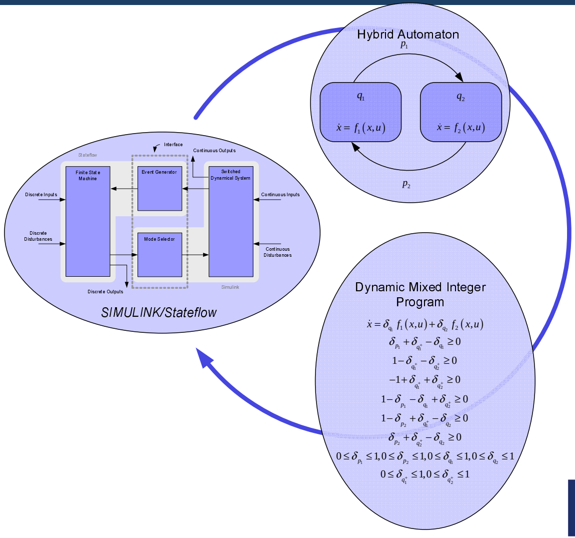
Outline

Optimal Control of Hybrid Systems

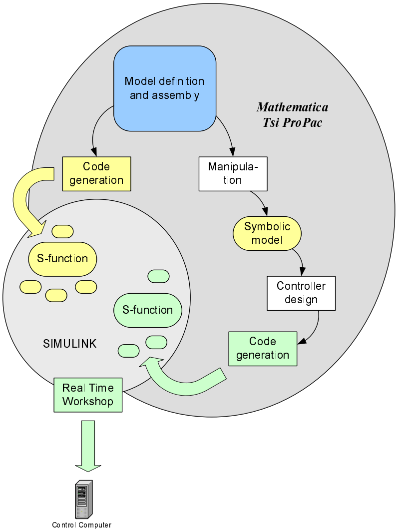
MODEL ENVIRONMENTS

- hybrid systems combine:
  - Discrete event system
  - Logical Constraints
  - Continuous nonlinear dynamics
- Simulink with Stateflow
  - Close to physical situation
  - Real-time implementation
- Hybrid Automaton
  - Compact, well-defined mathematical model
  - Primary theoretical tool
- Dynamic mixed-integer program
  - Excellent design model



Computational Tools

- Integrated symbolic and numerical computing
  - symbolic model assembly – aircraft, power systems, automotive, robotics
  - setup for numerical computation – e.g., NR, NRS
- Model transformation
  - Stateflow diagram to logical specification
  - Logical specification to IP formulas
  - Symbolic model to simulation model
- mixed-integer dynamic programming



Problem Definition

$$x_{k+1} = f\left(x_k, \delta_k, d_k, z_k, u_k\right), \quad k = 0, 1, \ldots, N-1$$
$$h\left(x_k, \delta_k, d_k, z_k, u_k, \delta_{k-1}, d_{k-1}, z_{k-1}\right) \leq 0$$

- x_k the continuous state (real numbers)
- δ_k the discrete state (mode) (binary or integer numbers)
- u_k the control, may be composed of discrete and continuous elements
- d_k discrete (binary) auxiliary variables
- z_k continuous (real) auxiliary variables

A control policy is a sequence of functions
$\pi = \{\mu_0\left(x_0, \delta_0\right), \mu_1\left(x_1, \delta_1\right), \ldots, \mu_{N-1}\left(x_{N-1}, \delta_N - 1\right)\}$, such that $u_k = \mu_k\left(x_k, \delta_k\right)$.
The optimal control problem is: given an initial state x_0, δ_0, is find a control policy that minimizes the cost function:

$$J_\pi\left(x_0, \delta_0\right) = g_N\left(x_N, \delta_N\right) + \sum_{k=0}^{N-1} g_k\left(x_k, \delta_k, \mu_k\left(x_k, \delta_k\right)\right)$$

Principle of Optimality

The optimal cost is

$$J^*(x_0, \delta_0) = \min_{\pi \in \Pi} J_\pi(x_0, \delta_0)$$

The optimal control policy μ^* is one that satisfies

$$J_{\pi^*}(x_0, \delta_0) \leq J_\pi(x_0, \delta_0) \quad \forall \pi \in \Pi$$

The Principle of Optimality: If $\pi^* = \left\{\mu_1^*, \ldots, \mu_{N-1}^*\right\}$ is an optimal policy, the the subpolicy ${\pi_i}^* = \left\{\mu_i^*, \ldots, \mu_{N-1}^*\right\}$, $1 \leq i \leq N-1$, is optimal with respect to the cost function

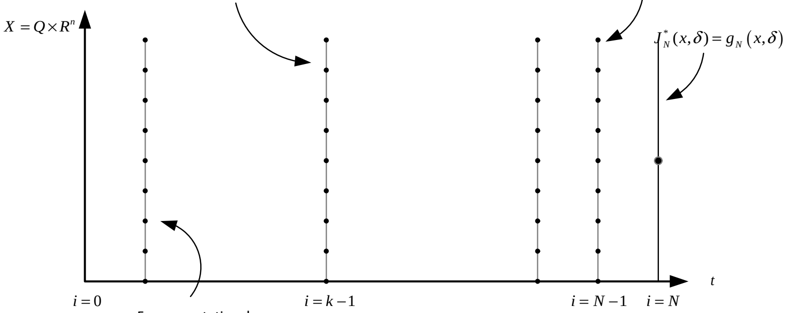
$$J_\pi(x_i, \delta_i) = g_N(x_N, \delta_N) + \sum_{k=i}^{N-1} g_k(x_k, \delta_k, \mu_k(x_k, \delta_k))$$

RECURSION

From each state at i=N-1 compute the optimal control for this stage. The optimization is carried out with constraints: mixed integer inequalities and dynamics.

$$J^*_{k-1}(x_{k-1},\delta_{k-1}) = \min_{\substack{u_{k-1}(x_{k-1},\delta_{k-1})\\ u_{k-1},x_k,\delta_{k-1}\in C}} \left\{ g_{k-1}(x_{k-1},\delta_{k-1},u_{k-1}) + J^*_k(x_k,\delta_k) \right\}$$

$$J^*_{N-1}(x_{N-1},\delta_{N-1}) = \min_{\substack{u_{N-1}(x_{N-1},\delta_{N-1})\\ u_{N-1},x_N,\delta_{N-1}\in C}} \left\{ g_{N-1}(x_{N-1},\delta_{N-1},u_{N-1}) + J^*_N(x_N,\delta_N) \right\}$$



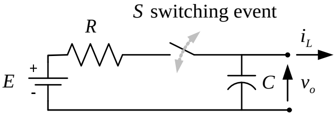
For computational purposes discretize the state space.

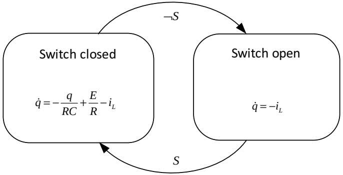
Algorithm

There are many inequality constraints so that determining feasible points is the essential computation. Moreover, most of the constraints are linear in binary variables.

1. Identify the binary and real variables and separate the inequalities into binary and real sets, binary equations contain only binary variables, real equations can contain both binary and real variables.
2. Use the Mathematica function Reduce to obtain all feasible solutions of the binary inequalities. Reduce is a very efficient solver, especially when the inequalities are linear although it is not limited to linear inequalities. In general, if there are N binary variables then there are 2^N combinations that need to be evaluated if one were to attempt to optimize by enumeration. But the feasible combinations are almost certainly much fewer.
3. Use Reduce to solve the real inequalities for the real variables for every feasible combination of binary variables. Many of these combinations of binary variables will not admit feasible real variables, so they can be dropped. The remaining combinations typically produce unique values for the real variables.
4. Enumerate the values of the cost for each feasible pair of binary and real variables and select the minimum.

Power Conditioning System Dynamics





$$\begin{aligned} \dot{q} &= -\frac{q}{RC} + \frac{E}{R} - i_L \quad \text{switchclosed} \\ \dot{q} &= -i_L \quad \text{switchopen} \end{aligned}$$

$$\begin{aligned} i &= \delta_{q_1} \left(\frac{E}{R} - \frac{q}{RC} \right) \\ v_0 &= \frac{q}{C} \end{aligned}$$

$$\dot{q} = \delta_{q_1} \left(-\frac{q}{RC} + \frac{E}{R} - i_L \right) + \delta_{q_2} \left(-i_L \right) \Rightarrow$$

$$q\left(t_{i+1}\right) = \delta_{q_1} \left(e^{-\frac{\Delta t}{RC}} q\left(t_i\right) + \left[1 - e^{-\frac{\Delta t}{RC}} \right] \left(CE - RCi_L\left(t_i\right) \right) \right) + \delta_{q_2} \left(q\left(t_i\right) - i_L\left(t_i\right) \Delta t \right)$$

Power Conditioning Logic

$$
\begin{aligned}
&(q_1(t) \oplus q_2(t)) \wedge (q_1(t^+) \oplus q_2(t^+)) \\
&\wedge (q_1(t) \wedge s \Rightarrow q_2(t^+)) \wedge (q_1(t) \wedge \neg s \Rightarrow q_1(t^+)) \\
&\wedge (q_2(t) \wedge s \Rightarrow q_1(t^+)) \wedge (q_2(t) \wedge \neg s \Rightarrow q_2(t^+))
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{gathered}
1 - \delta_{q_1^+} - \delta_{q_2^+} \geq 0 \\
-1 + \delta_{q_1^+} + \delta_{q_2^+} \geq 0 \\
1 - \delta_s - \delta_{q_1} + \delta_{q_2^+} \geq 0 \\
1 - \delta_s - \delta_{q_2} + \delta_{q_2^+} \geq 0 \\
\delta_s - \delta_{q_1} + \delta_{q_1^+} \geq 0 \\
\delta_s - \delta_{q_2} + \delta_{q_1^+} \geq 0 \\
0 \leq \delta_s \leq 1, 0 \leq \delta_{q_1} \leq 1, 0 \leq \delta_{q_2} \leq 1 \\
0 \leq \delta_{q_1^+} \leq 1, 0 \leq \delta_{q_2^+} \leq 1
\end{gathered}
$$

Optimization

- With $i_L = 0$ an equilibrium point is $\bar{q} = EC$.
- We wish to steer the system from the initial state q_0 to near $\bar{q}$ over the time interval $t \in [0, T]$ along a trajectory that minimizes

$$J = \alpha[q(T) - \bar{q}]^2 + \frac{1}{T}\int_0^T i^2\, dt$$

With $E = 1, C = 1, R = 1$, the equations reduce to

$$\begin{gathered} \dot{q} = \delta_{q_1}(-q + 1 - i_L) + \delta_{q_2}(-i_L) \\ i = \delta_{q_1}(1 - q) \\ v_0 = q \end{gathered}$$

In discrete time

$$q(t_{i+1}) = \delta_{q_1}\left(e^{-\Delta t} q(t_i) + \left[1 - e^{-\Delta t}\right](1 - i_L)\right) + (1 - \delta_{q_1})(q(t_i) - i_L \Delta t)$$

Tricks

Note that we can always write

$$\dot{q} = z$$

$$(q_1 \Rightarrow z = -q + 1 - i_L) \wedge (\neg q_1 \Rightarrow z = -i_L)$$

with $i_L = 0$ and bounds on capacitor charge $0 \leq q \leq 2$

$$\begin{array}{c} 1 - d_1 + z \geq 0 \\ 1 - 2d_2 + q + z \geq 0 \\ d_2 - \delta_{q_1} \geq 0 \\ -1 + d_1 + \delta_{q_1} \geq 0 \\ 1 - d_1 - z \geq 0 \\ 3 - 2d_2 - q - z \geq 0 \\ -1 \leq z \leq 1, 0 \leq d_1 \leq 1, 0 \leq d_2 \leq 1, \\ 0 \leq \delta_{q_1} \leq 1, 0 \leq q \leq 2 \end{array} \quad \Rightarrow \quad \begin{array}{c} z + \delta_{q_1} \geq 0 \\ 1 + q + z - 2\delta_{q_1} \geq 0 \\ \delta_{q_1} - z \geq 0 \\ 3 - 2\delta_{q_1} - q - z \geq 0 \\ -1 \leq z \leq 1, 0 \leq \delta_{q_1} \leq 1, \\ 0 \leq q \leq 2 \end{array}$$

Tricks

Similarly,we can set

$$i = w$$

and w satisfies

$$w + \delta_{q_1} \geq 0$$
$$1 + q + w - 2\delta_{q_1} \geq 0$$
$$\delta_{q_1} - w \geq 0$$
$$3 - 2\delta_{q_1} - q - w \geq 0$$
$$-1 \leq w \leq 1, 0 \leq \delta_{q_1} \leq 1, 0 \leq q \leq 2$$

Solution-1

$$J = \alpha[q_N - \bar{q}]^2 + \frac{1}{N}\sum_{i=0}^{N-1} w_i^2$$

with dynamics

$$q\left(i+1\right) = z\left(i\right)$$

$$\begin{gathered}
1 - d_2 + z \geq 0 \\
2.5592 - 2.7625d_1 - 0.2296q + z \leq 0 \\
d_1 - \delta_{q_1} \geq 0 \\
-1 + d_2 + \delta_{q_1} \geq 0 \\
-1 + d_2 + z \leq 0 \\
-1 + 0.7967d_1 - 0.7796q + z \leq 0 \\
-1 \leq z \leq 1, 0 \leq d_1 \leq 1, 0 \leq d_2 \leq 1, 0 \leq q \leq 2, 0 \leq \delta_{q_1} \leq 1
\end{gathered}$$

Solution 2

and transition logic

$$
\begin{gathered}
1-\delta_{q_1^+}-\delta_{q_2^+}\geq 0\\
-1+\delta_{q_1^+}+\delta_{q_2^+}\geq 0\\
1-\delta_s-\delta_{q_1}+\delta_{q_2^+}\geq 0\\
1-\delta_s-\delta_{q2}+\delta_{q_2^+}\geq 0\\
\delta_s-\delta_{q_1}+\delta_{q_1^+}\geq 0\\
\delta_s-\delta_{q_2}+\delta_{q_1^+}\geq 0\\
0\leq\delta_s\leq 1, 0\leq\delta_{q1}\leq 1, 0\leq\delta_{q2}\leq 1\\
0\leq\delta_{q_1^+}\leq 1, 0\leq\delta_{q_2^+}\leq 1
\end{gathered}
$$

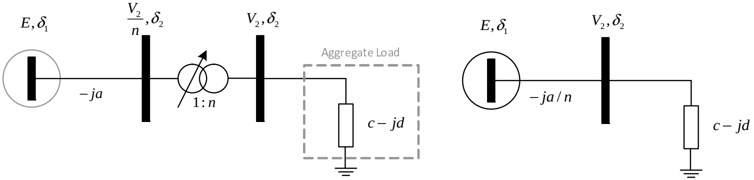
and current

$$
\begin{gathered}
w+\delta_{q1}\geq 0\\
1+q+w-2\delta_{q1}\geq 0\\
\delta_{q1}-w\geq 0\\
3-2\delta_{q1}-q-w\geq 0\\
-1\leq w\leq 1, 0\leq\delta_{q1}\leq 1, 0\leq q\leq 2
\end{gathered}
$$

Solution Summary

With zero load current, we get exactly as expected. For high current cost the switch remains open for all initial charge in the admissible range. For low current cost, the switch remains closed. With load, the switching strategy is more interesting although it still has a strong dependency on relative cost. For instance, with a current load of 0.1 amp, a time horizon of 2.5 sec, and terminal cost weight $\alpha = 0.25$, we obtain for $q \leq 1$ the switch is closed, for $q > 1$ the switch is open. If the weight is increased to $\alpha = 0.28$ the switch is closed for $q \leq 1$ and $q > 1.4$. It is open for $1 < q \leq 1.4$.

Simple Network Example – Setup



$$s_{k+1} = f\left(s_k, V_2, \eta\right)$$

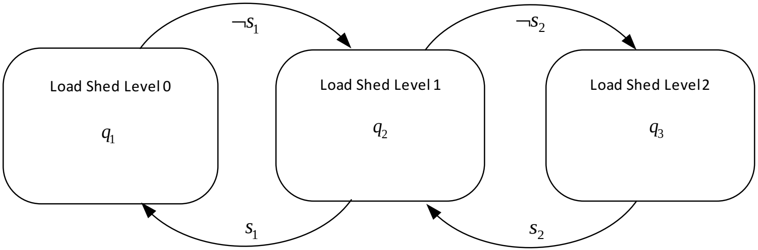
$$E = (1-\eta)\,\frac{\sqrt{{c_0}^2 + {d_0}^2}}{a/n}\,V_2$$

$$(V_2 = 1 \wedge 0 < E < 2) \vee (E = 2)$$

$$q_1 \Rightarrow \eta = 0, \quad q_2 \Rightarrow \eta = 0.4, \quad q_3 \Rightarrow \eta = 0.8$$

$$J = \sum_{k=0}^{N-1}\left(\|V_2(k) - 1\|^2 + r_1\|\eta_L(k)\|^2\right)$$

Simple Network Example – Transition Logic



Simple Network Example – IP Formulas 1

Voltage Control Logical Constraint:

$$3 - d_1 - E > 0, \quad 1 - d_1 + E > 0, \quad -2d_2 + E \geq 0$$
$$-2d_1 + V_2 \geq 0, \quad -2 + d_1 + V_2 \leq 0$$
$$0 \leq d_1, d_2 \leq 1, \quad 0 \leq E, V_2 \leq 2$$

Load shed parameter logic:

$$-0.4d_4 + \eta \geq 0, \quad -0.8d_5 + \eta \geq 0,$$
$$d_3 - \delta_{q_1^+} \geq 0, \quad d_4 - \delta_{q_2^+} \geq 0, \quad d_5 - \delta_{q_3^+} \geq 0$$
$$-1 + d_3 + \eta \leq 0, \quad -1 + 0.6d_4 + \eta \leq 0, \quad -1 + 0.2d_5 + \eta \leq 0$$
$$0 \leq d_3 \leq 1, \quad 0 \leq d_4 \leq 1, \quad 0 \leq d_5 \leq 1, \quad 0 \leq \eta \leq 1$$

Simple Network Example – IP Formulas 2

Transition logic:

$$1-\delta_{q_1}-\delta_{q_2}-\delta_{q_3}\geq 0,\quad -1+\delta_{q_1}+\delta_{q_2}+\delta_{q_3}\geq 0$$
$$1-\delta_{q_1^+}-\delta_{q_2^+}-\delta_{q_3^+}\geq 0,\quad -1+\delta_{q_1^+}+\delta_{q_2^+}+\delta_{q_3^+}\geq 0$$
$$1-\delta_{q1}+\delta_{q_1^+}-\delta_{s_1}\geq 0,\quad 1-\delta_{q2}+\delta_{q_1^+}-\delta_{s_1}\geq 0$$
$$1-\delta_{q2}+\delta_{q_2^+}-\delta_{s_2}\geq 0,\quad 1-\delta_{q3}+\delta_{q_2^+}-\delta_{s_2}\geq 0$$
$$-\delta_{q1}+\delta_{q_2^+}+\delta_{s_1}\geq 0$$
$$-\delta_{q_2}+\delta_{q_3^+}+\delta_{s_2}\geq 0,\quad -\delta_{q_3}+\delta_{q_3^+}+\delta_{s_2}\geq 0$$
$$0\leq\delta_{q_1}\leq 1, 0\leq\delta_{q2}\leq 1, 0\leq\delta_{q3}\leq 1$$
$$0\leq\delta_{q_1^+}\leq 1, 0\leq\delta_{q_2^+}\leq 1, 0\leq\delta_{q_3^+}\leq 1$$
$$0\leq\delta_{s_1}\leq 1, 0\leq\delta_{s_2}\leq 1$$

Simple Network Example