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A Canonical Parameterization of the Kronecker Form of a Matrix Pencil*

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The Kronecker form of a matrix pencil displays the invariant zero structure of a linear system. A minimal parameterization of the Kronecker form of a given pencil is derived to capture the invariant zero structure of any nearby pencil.

Key Words—Zeros; singularities; unfoldings; structural stability; bifurcation; versal deformation.

Abstract—The Kronecker form is the classical canonical form for matrix pencils under strict equivalence transformation. Consider a matrix pencil whose entries are smooth functions of a parameter vector. The Kronecker form of the parameterized pencil will, in general, be a discontinuous function of the parameters. For a linear time-invariant control system these discontinuities correspond to a change in the finite and infinite invariant zero structure. Since many control strategies require knowledge of, or place restrictions on, the zero structure, these points of discontinuity are of considerable interest. In this paper a general approach to the study of such points is developed in the framework of singularity theory. We derive a 'miniversal' parameterization of a given pencil. That is, a parameterized family of pencils that: (i) includes the given pencil, (ii) is locally equivalent to any other family up to a change of parameters, and (iii) uses the fewest number of parameters to achieve this property. All 'nearby' zero structures can be obtained by varying parameter values in the miniversal parameterization. From all miniversal parameterizations, one having a particularly uncluttered representation is selected as canonical. However, some restrictions on the finite elementary divisors are then required.

1. INTRODUCTION

Matrix pencils are matrix-valued functions of a scalar variable of the form $\mathbf{M}_1 + \mathbf{M}_2 s$. They are ubiquitous in the study of linear systems. The dynamic system $\dot{x} = \mathbf{A}x$ has associated pencil $s\mathbf{I} - \mathbf{A}$. The implicit control system $\mathbf{E}\dot{x} = \mathbf{A}x + \mathbf{B}u$, $y = \mathbf{C}x + \mathbf{D}u$ has associated system pencil

$$s\begin{bmatrix}\mathbf{E} & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}. \tag{1}$$

When $\mathbf{E} = \mathbf{I}$ the pencil corresponds to a linear state-space control system. Two pencils, $\mathbf{M}_1 + \mathbf{M}_2 s$ and $\mathbf{N}_1 + \mathbf{N}_2 s$, are said to be equivalent if there exist constant, invertible, matrices $\mathbf{P}$ and $\mathbf{Q}^{-1}$ such that $\mathbf{N}_1 = \mathbf{P}\mathbf{M}_1\mathbf{Q}^{-1}$ and $\mathbf{N}_2 = \mathbf{P}\mathbf{M}_2\mathbf{Q}^{-1}$. This relation is known as strict equivalence (s.e.; it is common to use $\mathbf{Q}$ in place of $\mathbf{Q}^{-1}$), and it can be used to establish a useful concept of equivalence for linear systems.

Many definitions of equivalence for linear systems have been proposed. Some of these require that the input–output behavior, or transfer matrix, be invariant (Rosenbrock, 1974; Fuhrmann, 1977; Verghese *et al.*, 1981; Pugh *et al.*, 1987; Kuijper and Schumacher, 1991). Others allow operations that change the input–output behavior (Kalman, 1972; Wonham and Morse, 1972; Morse, 1973; Thorp, 1973; Wolovich and Falb, 1976). This paper most closely follows the view of Thorp that state feedback, output-injection and coordinate transformations of the input, state and output spaces are allowable operations. As shown by Thorp, these correspond exactly to s.e. transformations on the corresponding pencil. The essential invariant under s.e. transformation is precisely the invariant zero structure (MacFarlane and Karcanias, 1976), which is a fundamental property of a control system. The invariance properties of equivalent control systems are clearly shown in the canonical Kronecker form (Gantmacher, 1959). Aling and Schumacher (1984) present a comprehensive discussion of

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zeros and their relation to the Kronecker invariants.

The question of uncertain parameters always arises when using mathematical representations of physical systems. This issue is particularly important when using the Jordan or Kronecker form. The eigenstructure or zero structure may depend discontinuously on system parameters, even when the matrices representing the system depend smoothly on those parameters. It is therefore of great interest to investigate the various structures that can arise from small perturbations of a nominal dynamical or control system.

Arnold (Arnold, 1981; Gilmore, 1981) examined this problem for square matrices. He considered the problem of a square matrix with entries that may depend holomorphically (that is, complex-differentiably) on parameters. In summary, any parameter-dependent family of square matrices for which the zero parameter value corresponds to a nominal matrix is called a deformation of the nominal matrix. Two deformations depending on the same parameters are equivalent if they are related by a near-identity similarity transformation, itself dependent on the same parameters. A deformation can be reparameterized by an analytic transformation of parameters. It is called versal if any other deformation is equivalent to it, possibly with reparameterization. A versal deformation contains all possible Jordan structures that can arise from a small perturbation. Note that at least one versal deformation is easy to come by. Simply add an independent parameter to every entry of the Jordan form of the nominal matrix. The versal deformation obtained by Arnold uses the smallest number of parameters possible. Arnold refers to versal deformations with this additional property as miniversal.

The theory presented by Arnold can be extended to other objects. Tannenbaum (1981) obtains results for the manifold of $(\mathbf{A}, \mathbf{B})$ matrix pairs under similarity, and an extension for $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ triples. This paper applies Arnold's program to implicit and state-space control systems, by considering matrix pencils under s.e. transformation. The results are most interesting when the parameterized object is in some sense structurally unstable or nongeneric. In the case of square matrices, this means repeated eigenvalues and nontrivial Jordan blocks. In the case of $(\mathbf{A}, \mathbf{B})$ pairs this means unreachable. The set of possible structural instabilities of a matrix pencil is extremely rich.

The sensitivity of the Kronecker form to perturbation clearly must be considered when developing stable numerical algorithms for computation. Van Dooren (1979, 1981) discusses such algorithms and their applications in detail. Demmel and Edelman (1993) build on the work of Van Dooren to calculate the codimension of the orbit of a matrix pencil. This is exactly the number of independent parameters in a miniversal deformation, independently confirming our own calculations. Elmroth and Kågström (1993) report an exhaustive study of 2×3 matrix pencils, including an experimental (numerical) study of how the generic and all 17 nongeneric Kronecker structures behave under random perturbations.

Section 2 summarizes the notation used in the remainder of the paper. Section 3 gives matrix pencils and s.e. transformation interpretations as manifolds, and uses the versality theorem to find an algorithm that will lead to a miniversal deformation. Section 4 describes the Kronecker form, and explains how using the Kronecker form simplifies calculation. Section 5 illustrates the computations on a simple pencil. Section 6 describes the miniversal deformation. The detailed calculation for one partition is given in the Appendix. Section 7 gives some nontrivial examples and Section 8 contains concluding remarks.

2. BACKGROUND AND NOTATION

It is convenient to represent both matrix pencils and strict equivalence transformations as matrix pairs, respectively $\boldsymbol{M} = (\mathbf{M}_1, \mathbf{M}_2) \stackrel{\text{def}}{=} \mathbf{M}_1 + \mathbf{M}_2 s$, and $\boldsymbol{G} = (\mathbf{G}_1, \mathbf{G}_2)$. The new pencil produced by operating on $\boldsymbol{M}$ with $\boldsymbol{G}$ is $(\mathbf{G}_1, \mathbf{M}_1\mathbf{G}_2^{-1}, \mathbf{G}_1\mathbf{M}_2\mathbf{G}_2^{-1})$. Let $\mathcal{M}$ be the set of all matrix pairs of given order, which throughout this paper are to be $m \times p$, i.e. $\mathcal{M} = \mathbb{C}^{m \times p} \times \mathbb{C}^{m \times p}$. The space of all s.e. transformations $\mathcal{G}$ on $\mathcal{M}$ is $\mathcal{G} = \mathrm{Gl}(m, \mathbb{C}) \times \mathrm{Gl}(p, \mathbb{C})$.

Scalar variables are indicated with italics or Greek letters, k, K or κ. Vectors and matrices are bold Roman characters, lower and upper case respectively, $\mathbf{x}$ and $\mathbf{A}$. A single element of a matrix is indicated using single brackets $[\mathbf{A}]_{ij} = \alpha_{ij}$. 0 denotes both the zero vector and the zero matrix. $\mathbf{0}$ is used in matrices as a 'space-filling' zero. The complex conjugate transpose of $\mathbf{A}$ is $\mathbf{A}^*$. Matrix pairs are written in bold italic, $\boldsymbol{M}$. The two members of a matrix pair are denoted by the same letter as the pencil itself, with subscripts indicating order, $\boldsymbol{M} \stackrel{\text{def}}{=} (\mathbf{M}_1, \mathbf{M}_2)$. By a space of matrix pairs is meant the set, $\{(\mathbf{M}_1, \mathbf{M}_2) : \mathbf{M}_1 \in \mathbb{C}^{i \times j}; \mathbf{M}_2 \in \mathbb{C}^{k \times l}\}$ for i, j, k, l specified constants. Elements of the matrix pair are indexed by three numbers, $m_{ijk} \stackrel{\text{def}}{=} [\boldsymbol{M}]_{ijk} \stackrel{\text{def}}{=} [\mathbf{M}_k]_{ij}$. The space is always given the structure of a vector space by

$(\mathbf{M}_1, \mathbf{M}_2) + (\mathbf{N}_1, \mathbf{N}_2) = (\mathbf{M}_1 + \mathbf{N}_1, \mathbf{M}_2 + \mathbf{N}_2)$ and $\lambda(\mathbf{M}_1, \mathbf{M}_2) = (\lambda\mathbf{M}_1, \lambda\mathbf{M}_2)$, $\lambda \in \mathbb{C}$. Finally, $(\mathbf{AM}_1\mathbf{B}, \mathbf{AM}_2\mathbf{B})$ is abbreviated by $\mathbf{AMB}$.

If a matrix is partitioned then double brackets are used to index the partitions. For example, to show that the ijth partition of $\mathbf{A}$ is zero, write $[\![\mathbf{A}]\!]_{ij} = 0$. Clearly the partition of a matrix pencil defines a partition of the matrix pair representing that pencil. If one term in a sum of pencils is partitioned then the same partitioning is applied to the other terms, and to the result. If a partitioned rectangular matrix is premultiplied by a square matrix then that square matrix is partitioned, rows and columns, like the rows of the rectangular matrix. Likewise, if a rectangular matrix is postmultiplied by a square matrix then the square matrix is partitioned, rows and columns, like the columns of the rectangular matrix. The extension to matrix pencils is clear.

$\mathbf{I}^{(n)}$ denotes an $n \times n$ identity matrix. $\mathbf{H}^{(n)}$ denotes an $n \times n$ matrix with ones on the first superdiagonal and zeros elsewhere. $\mathbf{H}_{(n)}$ denotes an $n \times n$ matrix with ones on the first subdiagonal and zeros elsewhere.

3. MANIFOLDS AND DEFORMATIONS

Many of the concepts and theorems required in this paper apply to the general situation of a smooth manifold acted upon by a Lie group. Only the details of the calculation are unique to matrix pencils under s.e. transformation.

3.1. *Lie groups, actions and orbits*

A Lie group is a group with a manifold structure. If $\mathcal{M}$ is a manifold and $\mathcal{G}$ is a Lie group with identity element $\boldsymbol{I}$, then a smooth map $\alpha: \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, also written $\alpha_{\boldsymbol{M}}(\boldsymbol{G})$ or $\boldsymbol{G} \cdot \boldsymbol{M}$, is an action of $\mathcal{G}$ on $\mathcal{M}$ if (i) $\boldsymbol{I} \cdot \boldsymbol{M} = \boldsymbol{M}$ and (ii) $\boldsymbol{G}_2 \cdot (\boldsymbol{G}_1 \cdot \boldsymbol{M}) = (\boldsymbol{G}_2 \cdot \boldsymbol{G}_1) \cdot \boldsymbol{M}$. Any action of a Lie group on a manifold defines an equivalence relation, as follows: $\boldsymbol{N}$ is equivalent to $\boldsymbol{M}$ if and only if there is a $\boldsymbol{G}$ such that $\boldsymbol{N} = \boldsymbol{G} \cdot \boldsymbol{M}$. The orbit of $\boldsymbol{M} \in \mathcal{M}$ under the action is every element of $\mathcal{M}$ equivalent to $\boldsymbol{M}$ written $\mathcal{O}_{\boldsymbol{M}} \stackrel{\text{def}}{=} \mathcal{G} \cdot \boldsymbol{M}$. The orbit of $\boldsymbol{M}$ is a submanifold of $\mathcal{M}$. For a proof see Varadarajan (1984).

3.2. *Deformations and versality*

This section very closely follows Arnold (1981). Consider a matrix pair $\boldsymbol{A} \in \mathcal{A}$, where $\mathcal{A}$ can be either $\mathcal{M}$ or $\mathcal{G}$, and a parameter vector $\mathbf{c} \in \mathcal{C} \subset \mathbb{C}^k$. A mapping $\boldsymbol{A}: \mathcal{C} \to \mathcal{A}$, written $\boldsymbol{A}(\mathbf{c})$, is called a deformation of $\boldsymbol{A}$ if, (i) the entries of $\boldsymbol{A}(\mathbf{c})$ are power series in the elements of $\mathbf{c}$, convergent in some neighborhood of $\mathbf{c} = 0$, and (ii) $\boldsymbol{A}(0) = \boldsymbol{A}$. The space $\mathcal{C}$ is called the base of the deformation. Now consider $\mathcal{M}$ and $\mathcal{G}$, and the action of $\mathcal{G}$ on $\mathcal{M}$. Two deformations of the same element of $\mathcal{M}$, $\boldsymbol{M}(\mathbf{c})$ and $\boldsymbol{N}(\mathbf{c})$ say, with the same base, are equivalent if there exists a deformation of the identity element of $\mathcal{G}\boldsymbol{G}(\mathbf{c})$, with that base such that $\boldsymbol{M}(\mathbf{c}) = \boldsymbol{G}(\mathbf{c}) \cdot \boldsymbol{N}(\mathbf{c})$. Note that $\boldsymbol{M}(0) = \boldsymbol{N}(0) = \boldsymbol{M}$ and $\boldsymbol{G}(0) = \boldsymbol{I}$.

Next consider a second parameter vector $\mathbf{d} \in \mathcal{D} \subset \mathbb{C}^l$, and a mapping $\phi: \mathcal{D} \to \mathcal{C}$. Require that (i) the elements of ϕ be power series in the elements of $\mathbf{c}$ convergent in some neighborhood of $\mathbf{c} = 0$, and (ii) $\phi(0) = 0$. Then define the composition $\boldsymbol{M}(\phi(d))$, the mapping induced by $\boldsymbol{M}(\mathbf{c})$ under ϕ.

Definition. A deformation $\boldsymbol{M}(\mathbf{c})$is called versal if any arbitrary deformation of the same pencil is equivalent to a deformation induced by $\boldsymbol{M}(\mathbf{c})$. That is, $\boldsymbol{N}(\mathbf{d})$ can be written $\boldsymbol{N}(\mathbf{d}) = \boldsymbol{G}(\mathbf{d}) \cdot \boldsymbol{M}(\phi(\mathbf{d}))$ with $\boldsymbol{M}(0) = \boldsymbol{N}(0) = \boldsymbol{M}$, $\boldsymbol{G}(0) = \boldsymbol{I}$, and $\phi(0) = 0$. A versal deformation $\boldsymbol{M}(\mathbf{c})$ is called miniversal if, for every versal deformation $\boldsymbol{N}(d)$, $\dim \mathcal{C} \leq \dim \mathcal{D}$. □

A deformation $\boldsymbol{M}(\mathbf{c})$ defines a smooth submanifold in a neighborhood of $\boldsymbol{M}$, namely $\boldsymbol{M}(\mathcal{C})$. One can test for the versality of the deformation using properties of this manifold, through the following definition:

Definition. Let $\mathcal{N} \subset \mathcal{M}$ be a smooth submanifold. Consider a smooth mapping $\boldsymbol{M}: \mathcal{C} \to \mathcal{M}$ of a manifold $\mathcal{C}$ into $\mathcal{M}$, and let $\mathbf{c} \in \mathcal{C}$ satisfy $\boldsymbol{M}(\mathbf{c}) \in \mathcal{N}$. Then the map $\boldsymbol{M}$ is said to be transversal to $\mathcal{N}$ at $\mathbf{c}$, or to intersect $\mathcal{N}$ transversally at $\mathbf{c}$, if

$$\mathrm{T}_{\boldsymbol{M}(\mathbf{c})}\mathcal{M} = \mathrm{d}\boldsymbol{M}(\mathrm{T}_{\mathbf{c}}\mathcal{C}) + \mathrm{T}_{\boldsymbol{M}(\mathbf{c})}\mathcal{N}. \qquad \square$$

Here $\mathrm{T}_{\boldsymbol{A}}\mathcal{A}$ denotes the tangent space to manifold $\mathcal{A}$ at the point $\boldsymbol{A}$ and $\mathrm{d}\boldsymbol{M}: \mathrm{T}_{\mathbf{c}}\mathcal{C} \to \mathrm{T}_{\boldsymbol{M}(\mathbf{c})}\mathcal{M}$ is the differential of the mapping $\boldsymbol{M}\mathcal{C} \to \mathcal{M}$. Recall that $\mathrm{T}_{\boldsymbol{A}}\mathcal{A}$ is a vector space with dimension equal to the dimension of $\mathcal{A}$ at $\boldsymbol{A}$. The versality theorem relates versal deformations to transversal interactions:

Versality Theorem. A deformation $\boldsymbol{M}(\mathbf{c})$ of a matrix pencil $\boldsymbol{M}$ is versal if and only if the mapping $\boldsymbol{M}: \mathcal{C} \to \mathcal{M}$ is transversal to the orbit of $\boldsymbol{M}$ at $\mathbf{c} = 0$.

Proof. See Tannenbaum (1981). □

The versality theorem states that a smooth mapping $\boldsymbol{M}: \mathcal{C} \to \mathcal{M}$ satisfying $M(0) = \boldsymbol{M}$ with $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} + \mathrm{d}\boldsymbol{M}(\mathrm{T}_0\mathcal{C}) = \mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ will be a versal deformation of $\boldsymbol{M}$. If in addition the dimension of $\mathcal{C}$ is minimal, the mapping is a miniversal deforma-

tion. The proof of the versality theorem uses the inverse function theorem. Therefore, the versality of the deformation is guaranteed only in a neighborhood of $\boldsymbol{M}$.

Given a manifold $\mathcal{M}$, a Lie group $\mathcal{G}$, an action $\alpha: \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ denoted by $\alpha_{\boldsymbol{M}}(\boldsymbol{G}) = \boldsymbol{G} \cdot \boldsymbol{M}$, and an inner product on $\mathcal{M}$, $\langle \boldsymbol{M}, \boldsymbol{N} \rangle \mapsto \mathbb{R}$, consider the following procedure:

(1) Denote an element of $\mathcal{M}$ that is of special interst by $\boldsymbol{M}$. The orbit of $\boldsymbol{M}$ is $\mathcal{O}_{\boldsymbol{M}} \stackrel{\text{def}}{=} \mathcal{G} \cdot \boldsymbol{M}$. By definition the mapping $\alpha_{\boldsymbol{M}}$ is form $\mathcal{G}$ onto $\mathcal{O}_{\boldsymbol{M}}$.
(2) Find the differential of $\alpha_{\boldsymbol{M}}$ at $\mathcal{G} = \boldsymbol{I}$. This is a mapping $\mathrm{d}\alpha_{\boldsymbol{M}}: \mathrm{T}_{\boldsymbol{I}}\mathcal{G} \to \mathrm{T}_{\boldsymbol{M}}\mathcal{M}$. The rank of $\mathrm{d}\alpha_{\boldsymbol{M}}$ is equal to the rank of $\alpha_{\boldsymbol{M}}$ (by definition). As a consequence, since α is onto the mapping $\mathrm{d}\alpha_{\boldsymbol{M}}: \mathrm{T}_{\boldsymbol{I}}\mathcal{G} \to \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$, where $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$ is a linear vector subspace of $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$, is also onto (Boothby, 1986). Therefore, the tangent space to the orbit of $\boldsymbol{M}$ can be characterized as $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} = \{\mathrm{d}_{\alpha \boldsymbol{M}}(\boldsymbol{V}^{\mathcal{G}}): \text{for all } \boldsymbol{V}^{\mathcal{G}} \in \mathrm{T}_{\mathbf{I}}\mathcal{G}\}$.
(3) Using the results of step 2, find all members of $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ orthogonal to $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$. This set, $(\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^{\perp} = \{\boldsymbol{V}^{\mathcal{M}}: \langle \boldsymbol{V}^{\mathcal{M}}, \boldsymbol{V}^{\mathcal{O}} \rangle = 0 \text{ for all } \boldsymbol{V}^{\mathcal{O}} \in \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}\}$. Although the inner product was defined for $\mathcal{M}$, members of $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ can be considered as elements of $\mathcal{M}$. Alternatively, the inner product can be defined on $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ instead of $\mathcal{M}$. The set $(\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^{\perp}$ is a vector subspace of $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$. A standard theorem on inner product spaces (Hoffman and Kunze, 1971) gives $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} \oplus (\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^{\perp} = \mathrm{T}_{\boldsymbol{M}}\mathcal{M}$.
(4) Form a basis of pencils, $\{\boldsymbol{U}_i\}$, spanning $(\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^{\perp}$. Denote the dimension of $(\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^{\perp}$ by k.
(5) Simplify the structure of each $\boldsymbol{U}_i$ by adding pencils that lie in $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$. The resulting set, $\{\boldsymbol{V}_i\}$, is still linearly independent, but no longer spans $(\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^{\perp}$. The critical property that is preserved is $\mathrm{span}\{\boldsymbol{V}_i\} \oplus \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} = \mathrm{T}_{\boldsymbol{M}}\mathcal{M}$. The set $\{\boldsymbol{V}_i\}$ is simple if each $\boldsymbol{V}_i$ has only one nonzero entry.
(6) Construct a mapping from $\mathscr{C}$ to $\mathcal{M}$ as follows:

$$\boldsymbol{M}(\mathbf{c}) = \boldsymbol{M} + \sum_{n=1}^{k} c_n \boldsymbol{V}_n, \tag{2}$$

where $(c_1, c_2, \ldots, c_k) = \mathbf{c} \in \mathscr{C}$. Implicit in this construction is an identification between $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ and $\mathcal{M}$, the use of the identity map as a coordinate function on $\mathscr{C}$, and the fact that $\mathcal{M}$ is a vector space over $\mathbb{C}$. The proof presented here also assumes that $\mathcal{M}$ is given a manifold structure by selecting its components, as in a matrix or matrix pencil.

Claim. The mapping $\boldsymbol{M}(\mathbf{c})$ is a miniversal deformation of $\boldsymbol{M}$.

Proof. Clearly $\boldsymbol{M}(0) = \boldsymbol{M}$. Because $\boldsymbol{M}$ and the k pencils $\boldsymbol{V}_i$ are constant, $\boldsymbol{M}(\mathbf{c})$ is an analytic function of $\mathbf{c}$. So $\boldsymbol{M}(\mathbf{c})$ is a deformation of $\boldsymbol{M}$.

Now show versality. For the following let the coordinate function on $\mathcal{M}$, $\phi: \mathcal{M} \to \mathbb{C}^l$, be an indexing function. For example, in the case of $m \times p$ matrix pencils, let $\phi_j(\boldsymbol{M}) = [\boldsymbol{M}]_{\mu(j)}$, where μ is an isomorphism between the set of triplets $\{rst: 1 \le r \le m,\ 1 \le s \le p,\ 1 \le t \le 2\}$ and the first $2mp$ natural numbers. Let $\{\boldsymbol{E}_j^{\boldsymbol{M}}\}$ be the natural basis for $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ and let $\{\boldsymbol{E}_i^{\boldsymbol{0}}\}$ be the natural basis for $\mathrm{T}_{\mathbf{0}}\mathscr{C}$. Set $(y_1, y_2, \ldots, y_l) \in \mathbb{C}^l$ and $(c_1, c_2, \ldots, c_k) \in \mathbb{C}^k$. The expression for $\boldsymbol{M}(\mathbf{c})$ in local coordinates is

$$y_j = \phi_j\left(\boldsymbol{M} + \sum_{n=1}^{k} c_n \boldsymbol{V}_n\right), \tag{3}$$

$$y_j = [\boldsymbol{M}]_{\mu(j)} + \sum_{n=1}^{k} c_n [\boldsymbol{V}_n]_{\mu(j)}, \tag{4}$$

and so

$$\frac{\partial y_j}{\partial c_i} = [\boldsymbol{V}_i]_{\mu(j)}. \tag{5}$$

Then (Boothby, 1986)

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{E}_i^{\boldsymbol{0}}) = \sum_{j=1}^{l} \left(\frac{\partial y_j}{\partial c_i}\right)_{\boldsymbol{M}} \boldsymbol{E}_j^{\mathbf{M}} = \sum_{j=1}^{l} [\boldsymbol{V}_i]_{\mu(j)} \boldsymbol{E}_j^{\mathbf{M}} = \boldsymbol{V}_i. \tag{6}$$

So, $\mathrm{d}\boldsymbol{M}(\mathrm{T}_{\mathbf{0}}\mathscr{C}) = \mathrm{span}\{\boldsymbol{V}_i\}$. By the construction then

$$\mathrm{d}\boldsymbol{M}(\mathrm{T}_{\mathbf{0}}\mathscr{C}) + \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} = \mathrm{T}_{\boldsymbol{M}}\mathcal{M}, \tag{7}$$

so $\boldsymbol{M}(\mathbf{c})$ intersects $\mathcal{O}_{\boldsymbol{M}}$ transversally at $\boldsymbol{M}$. Therefore, by the versality theorem, $\boldsymbol{M}(\mathbf{c})$ is a versal deformation. Finally, by the construction, the sum in the transversality equation is direct, and so no parameter space of smaller dimension can satisfy the transversality requirement. Therefore, $\boldsymbol{M}(\mathbf{c})$ is a miniversal deformation. ■

The following sections carry out these steps. The linear transformations in step 5 are used to give the deformation a particular structure. Specifically, it is desirable to distinguish between perturbations of tlhe linear part of the pencil, called noncausal perturbations, and perturbations to the constant part of the pencil, called causal perturbations. Ideally, the final parameterization is separated into purely causal and purely noncausual parts. Clearly if the results is simple then this separation is achieved.

3.3. *Manifold representation of system pencils and strict equivalence*

$\mathcal{M}$ and $\mathcal{G}$ are given the structure of complex analytic manifolds by associating each element

with a distinct coordinate in $\mathbb{C}^{2mp}$ or $\mathbb{C}^{m^2+p^2}$, respectively. $\mathcal{G}$ inherits the structure of a group from Gl, with multiplication defined in the natural way, $\boldsymbol{GH} = (\mathbf{G}_1, \mathbf{G}_2)(\mathbf{H}_1, \mathbf{H}_2) \stackrel{\text{def}}{=} (\mathbf{G}_1\mathbf{H}_1, \mathbf{G}_2\mathbf{H}_2)$, with identity $\boldsymbol{I} = (\mathbf{I}^{(m)}, \mathbf{I}^{(p)})$. Then $\mathcal{G}$ is a (complex) Lie group of dimension $m^2 + p^2$, and the map $\boldsymbol{G} \cdot \boldsymbol{M} \stackrel{\text{def}}{=} (\mathbf{G}_1\mathbf{M}_1\mathbf{G}_2^{-1}, \mathbf{G}_1\mathbf{M}_2\mathbf{G}_2^{-1})$ is an action of $\mathcal{G}$ on $\mathcal{M}$. The equivalence relation induced by this action is strict equivalence.

3.4. *The tangent spaces*

One can treat tangent vectors to $\mathcal{M}$ at $\boldsymbol{M}$ as elements of $\mathbb{C}^{m\times p} \times \mathbb{C}^{m\times p}$, and consider a basis for $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ to be $\{\boldsymbol{E}^{\mathbf{M}}_{ijk}$ for $i \in 1, \ldots, m;\ j \in 1, \ldots, p;\ k \in 1, 2\}$, where $\boldsymbol{E}^{\mathbf{M}}_{ijk} \in \mathbb{C}^{m\times p} \times \mathbb{C}^{m\times p}$ is defined by $[\boldsymbol{E}^{\mathbf{M}}_{ijk}]_{rst} = \delta_{ir}\delta_{js}\delta_{kt}$, that is, $\boldsymbol{E}^{\mathbf{M}}_{ijk}$ is the matrix pair with a one in the ijk element and zeros everywhere else.

In exactly the same way, treat tangent vectors to $\mathcal{G}$ at $\mathbf{G}$ as elements of $\mathbb{C}^{m\times m} \times \mathbb{C}^{p\times p}$. Use as a basis for $\mathrm{T}_{\boldsymbol{G}}\mathcal{G}$ the set $\{\boldsymbol{E}^{\mathbf{G}}_{ijk}$ for $(i, j \in 1, \ldots, m; k = 1)$ or $(i, j \in 1, \ldots, p; k = 2)\}$. Note that $\boldsymbol{E}^{\mathbf{G}}_{ijk}$ is not an element of $\mathrm{Gl}(m) \times \mathrm{Gl}(p)$.

Consider the differential of $\alpha_{\boldsymbol{M}} = \mathbf{G}_1\mathbf{M}\mathbf{G}_2^{-1}$. Only the point $\boldsymbol{G} = \boldsymbol{I}$ is of interest. Then, if $\boldsymbol{V}^{\mathcal{G}} = (\mathbf{V}_1^{\mathcal{G}}, \mathbf{V}_2^{\mathcal{G}}) \in \mathrm{T}_{\boldsymbol{G}}\mathcal{G}$, the differential is

$$\begin{aligned} \mathrm{d}\alpha_{\boldsymbol{M}}(\boldsymbol{V}^{\mathcal{G}}) &= \mathbf{V}_1^{\mathcal{G}}\boldsymbol{M}\mathbf{G}_2^{-1} - \mathbf{G}_1\boldsymbol{M}\mathbf{G}_2^{-1}\mathbf{V}_2^{\mathcal{G}}\mathbf{G}_2^{-1} \\ &= \mathbf{V}_1^{\mathcal{G}}\boldsymbol{M} - \boldsymbol{M}\mathbf{V}_2^{\mathcal{G}}. \end{aligned} \tag{8}$$

Use the fact that the restriction $\mathrm{d}\alpha_{\boldsymbol{M}}: \mathrm{T}_{\boldsymbol{I}}\mathcal{G} \to \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$ is onto to characterize the tangent space to the orbit of $\boldsymbol{M}$:

$$\begin{aligned} \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} &= \{\mathrm{d}\alpha_{\boldsymbol{M}}(\boldsymbol{V}^{\mathcal{G}}) : \boldsymbol{V}^{\mathcal{G}} \in \mathrm{T}_{\boldsymbol{I}}\mathcal{G}\} \\ &= \{\mathbf{V}_1^{\mathcal{G}}\boldsymbol{M} - \boldsymbol{M}\mathbf{V}_2^{\mathcal{G}} : \boldsymbol{V}^{\mathcal{G}} \in \mathrm{T}_{\boldsymbol{I}}\mathcal{G}\}. \end{aligned} \tag{9}$$

Recall that the tangent vectors to a manifold of matrix pairs are themselves represented by matrix pairs. Then $\mathrm{T}_{\boldsymbol{M}}\mathcal{M} = \mathbb{C}^{m\times p} \times \mathbb{C}^{m\times p}$, $\mathrm{T}_{\boldsymbol{I}}\mathcal{G} = \mathbb{C}^{m\times m} \times \mathbb{C}^{p\times p}$, and $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$ is a subspace of $\mathbb{C}^{m\times p} \times \mathbb{C}^{m\times p}$.

3.5. *Pencils orthogonal to* $\mathrm{T}_{\mathbf{M}}\mathcal{O}_{\mathbf{M}}$

Now describe all tangent vectors orthogonal to the orbit of $\boldsymbol{M}$, i.e. all $\boldsymbol{U} \in \mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ such that $\boldsymbol{V} \in \mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}} \Rightarrow \langle \boldsymbol{U}, \boldsymbol{V}\rangle = 0$. A suitable inner product is defined by

$$\langle \boldsymbol{U}, \boldsymbol{V}\rangle \stackrel{\text{def}}{=} \mathrm{tr}\{\mathbf{U}_1^*\mathbf{V}_1\} + \mathrm{tr}\{\mathbf{U}_2^*\mathbf{V}_2\}. \tag{10}$$

Using properties of the trace, and substituting $\mathbf{V}_1^{\mathcal{G}}\boldsymbol{M} - \boldsymbol{M}\mathbf{V}_2^{\mathcal{G}}$ for $\boldsymbol{V}$:

$$\begin{aligned} \langle \boldsymbol{U}, \boldsymbol{V}\rangle = \mathrm{tr}\{(\mathbf{M}_1\mathbf{U}_1^* + \mathbf{M}_2\mathbf{U}_2^*)\mathbf{V}_1^{\mathcal{G}}\} \\ - \mathrm{tr}\{(\mathbf{U}_1^*\mathbf{M}_1 + \mathbf{U}_2^*\mathbf{M}_2)\mathbf{V}_2^{\mathcal{G}}\}. \end{aligned} \tag{11}$$

So $\langle \boldsymbol{U}, \boldsymbol{V}\rangle$ is zero if and only if this final expression is zero. Recall that $\boldsymbol{V}^{\mathcal{G}}$ is completely arbitrary. Therefore, the orthogonality conditions on $\boldsymbol{U}$ are

$$\mathbf{M}_1\mathbf{U}_1^* + \mathbf{M}_2\mathbf{U}_2^* = 0, \tag{12a}$$

$$\mathbf{U}_1^*\mathbf{M}_1 + \mathbf{U}_2^*\mathbf{M}_2 = 0. \tag{12b}$$

All the pieces are now in place to calculate a miniversal deformation for matrix pencils under s.e. transformation. The actual algebra is made much easier by considering only matrix pencils in the Kronecker canonical form. In particular, this allows calculation on a partition by partition basis, rather than on the entire pencil all at once.

4. THE KRONECKER FORM

The Kronecker form is uniquely determined by the following set of invariants:

Kronecker column (or right) indices

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_h = 0 < \varepsilon_{h+1} \leq \varepsilon_{h+2} \leq \cdots \leq \varepsilon_p;$$

Kronecker row (or left) indices

$$\eta_1 = \eta_2 = \cdots = \eta_g = 0 < \eta_{g+1} \leq \eta_{g+2} \leq \cdots \leq \eta_q;$$

degree of infinite divisors

$$\rho_1 = \rho_2 = \cdots = \rho_r = 1 < \rho_{r+1} \leq \rho_{r+2} \leq \cdots \leq \rho_v;$$

a square matrix $\mathbf{J}$ in Jordan normal form containing the finite zero structure.

A matrix pencil in Kronecker form has structure $\boldsymbol{M}(s) = \mathrm{diag}\{\mathbf{0}, \boldsymbol{L}_{\varepsilon_{h-1}}(s), \ldots, \boldsymbol{L}_{\varepsilon_p}(s), \boldsymbol{L}^{\mathrm{T}}_{\eta_{g-1}}(s), \ldots, \boldsymbol{L}^{\mathrm{T}}_{\eta_q}(s), \boldsymbol{H}^{(\rho_1)}(s), \ldots, \boldsymbol{H}^{(\rho_r)}(s), \mathbf{J} + s\mathbf{I}^{(n)}\}$ where $\mathbf{0}$ is a $g \times h$ zero pencil, and $\boldsymbol{L}_l(s) = [0\ \mathbf{I}^{(l)}] + [\mathbf{I}^{(l)}\ 0]s$. The blocks are gathered into partitions, themselves block diagonal, so, $\boldsymbol{M}(s) = \mathrm{diag}\{\boldsymbol{M}_{\mathbf{z}}(s), \boldsymbol{M}_{\boldsymbol{\epsilon}}(s), \boldsymbol{M}_{\boldsymbol{\eta}}(s), \boldsymbol{M}_{\infty}(s), \boldsymbol{M}_{\mathbf{f}}(s)\}$. Here $\boldsymbol{M}_{\mathbf{z}}(s) = \mathbf{0}$, $\boldsymbol{M}_{\boldsymbol{\epsilon}}(s) = \mathrm{diag}\{\boldsymbol{L}_{\varepsilon_{h-1}}(s), \ldots, \boldsymbol{L}_{\varepsilon_p}(s)\}$, $\boldsymbol{M}_{\boldsymbol{\eta}}(s) = \mathrm{diag}\{\boldsymbol{L}^{\mathrm{T}}_{\eta_{g-1}}(s), \ldots, \boldsymbol{L}^{\mathrm{T}}_{\eta_q}(s)\}$, $\boldsymbol{M}_{\infty}(s) = \mathrm{diag}\{\boldsymbol{H}^{(\rho_1)}(s)\}, \ldots, \boldsymbol{H}^{(\rho_r)}(s)\}$ and $\boldsymbol{M}_{\mathbf{f}}(s) = \mathbf{J} + s\mathbf{I}^{(n)}$.

Let $\boldsymbol{N}$ be a matrix pencil and let $\boldsymbol{M} = \mathbf{S}\boldsymbol{N}\mathbf{T}^{-1}$ be in Kronecker form. Assume $\boldsymbol{U}$ satisfies the orthogonality conditions for $\boldsymbol{M}$. Form $\boldsymbol{V} = \mathbf{S}^*\boldsymbol{U}\mathbf{T}^{-*}$. Then,

$$\mathbf{N}_1\mathbf{V}_1^* + \mathbf{N}_2\mathbf{V}_2^* = \mathbf{0}, \tag{13a}$$

$$\mathbf{V}_1^*\mathbf{N}_1 + \mathbf{V}_2^*\mathbf{N}_2 = \mathbf{0}. \tag{13b}$$

So in general one can put a pencil into a canonical form, find a set of orthogonal pencils and transform back to the original form, if desired. Thus, it involves no loss of generality to assume that $\boldsymbol{M}$ is a matrix pencil in Kronecker form.

Because the Kronecker form is block diagonal, the orthogonality equations can be solved indepndently for each partition of $\boldsymbol{U}$. The

partition form of the orthogonality equations is

$$[\![\mathbf{M}_1]\!]_{ji}[\![\mathbf{U}_1^*]\!]_{ji} + [\![\mathbf{M}_2]\!]_{ji}[\![\mathbf{U}_2^*]\!]_{ji} = 0, \quad (14a)$$

$$[\![\mathbf{U}_1^*]\!]_{ji}[\![\mathbf{M}_1]\!]_{ii} + [\![\mathbf{U}_2^*]\!]_{ji}[\![\mathbf{M}_2]\!]_{ii} = 0, \quad (14b)$$

where $\boldsymbol{U}$ is partitioned following $\boldsymbol{M}$.

Also, exploit the block diagonal structure of the Kronecker form in step 5 of the procedure to modify $\boldsymbol{U}$ partition by partition. Do this by generating vectors in $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$ that are zero everywhere except the partition in question, and adding them to $\boldsymbol{U}$. Generate such a vector, say $\boldsymbol{V}^{\mathcal{O}}$, as follows.

Partition $\mathbf{V}_1^{\mathcal{G}}$, rows and columns, like the rows of $\boldsymbol{M}$. Partition $\mathbf{V}_2^{\mathcal{G}}$, rows and columns, like the columns of $\boldsymbol{M}$. (Recall that $\mathbf{V}_1^{\mathcal{G}}$ and $\mathbf{V}_2^{\mathcal{G}}$ are square.) Then,

$$[\![\mathbf{V}_1^{\mathcal{O}}]\!]_{ij} = [\![\mathbf{V}_1^{\mathcal{G}}\mathbf{M}_1 - \mathbf{M}_1\mathbf{V}_2^{\mathcal{G}}]\!]_{ij}$$
$$= [\![\mathbf{V}_1^{\mathcal{G}}]\!]_{ji}[\![\mathbf{M}_1]\!]_{jj} - [\![\mathbf{M}_1]\!]_{ii}[\![\mathbf{V}_2^{\mathcal{G}}]\!]_{ij}, \quad (15a)$$

$$[\![\mathbf{V}_2^{\mathcal{O}}]\!]_{ij} = [\![\mathbf{V}_1^{\mathcal{G}}\mathbf{M}_2 - \mathbf{M}_2\mathbf{V}_2^{\mathcal{G}}]\!]_{ij}$$
$$= [\![\mathbf{V}_1^{\mathcal{G}}]\!]_{ij} - [\![\mathbf{M}_2]\!]_{jj} - [\![\mathbf{M}_2]\!]_{ii}[\![\mathbf{V}_2^{\mathcal{G}}]\!]_{ij}. \quad (15b)$$

Since $\mathbf{V}^{\mathcal{G}}$ is arbitrary, restrict all partitions except $[\![\mathbf{V}_1^{\mathcal{G}}]\!]_{ij}$ and $[\![\mathbf{V}_2^{\mathcal{G}}]\!]_{ij}$ to be zero, and select these nonzero partitions in any convenient way. Note that $\mathbf{V}_1$ and $\mathbf{V}_2$ are square but the (off-diagonal) partitions will not be, in general.

5. A SIMPLE EXAMPLE

Consider the following pencil in Kronecker form:

$$\boldsymbol{M}(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{M} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right).$$

Vectors in the tangent space to the orbit at $\boldsymbol{M}$, $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$ have the form

$$\boldsymbol{V}^{\mathcal{O}} = \left(\mathbf{S} - \mathbf{T}, \mathbf{S}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{T} \right),$$

where $\mathbf{S}$ and $\mathbf{T}$ are arbitrary 2×2 matrices. The orthogonality conditions show that all tangent vectors in $\mathrm{T}_{\boldsymbol{M}}\mathcal{M}$ orthogonal to the tangent space $\mathrm{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}}$ are of the following form:

$$\boldsymbol{U} = \left(\begin{bmatrix} 0 & 0 \\ -c_1 & 0 \end{bmatrix}, \begin{bmatrix} c_1 & 0 \\ c_2 & c_1 \end{bmatrix} \right).$$

The deformation corresponding to c_2 is simple and strictly noncausal, as desired, but the deformation corresponding to c_1 is neither. Setting

$$\mathbf{S} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

the pencil

$$\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

is tangent to the orbit of $\boldsymbol{M}$, so form

$$\left(\begin{bmatrix} 0 & 0 \\ -c_1 & 0 \end{bmatrix}, \begin{bmatrix} c_1 & 0 \\ c_2 & c_1 \end{bmatrix} \right) - c_1 \left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

and redefine $c_1 = -c_2/3$. The result is the canonical deformation for this pencil:

$$\left(\begin{bmatrix} 0 & 0 \\ c_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ c_2 & 0 \end{bmatrix} \right),$$

which is both simple and composed only of causual and purely noncausal elements. The parameterized pencil is then

$$\left(\begin{bmatrix} 1 & 0 \\ c_1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ c_2 & 0 \end{bmatrix} \right) \Rightarrow \begin{bmatrix} 1 & s \\ c_1 + c_2 s & 1 \end{bmatrix}.$$

In this case, if the perturbed system must be causal, c_2 is constrained to be 0.

6. A CANONICAL DEFORMATION

This section describes a simple miniversal deformation of the Kronecker form. There is, however, a restriction in order to ensure that each parameter appears only once. This is as follows.

Restriction R. The finite zero generalized eigenvalues of the pencil must be distinct.

The canonical parameterization was developed with applications in mind (relative degree, regulation of parameter-dependent systems) that stress zeros at infinity, zeros at the origin and the singular parts of the pencil. No restriction is placed on these structures. Almost every aspect of the parameterization has been carried out for the completely general case, except the final simplification of the structure. In particular, the number of independent parameters given for each partition is valid in general. For a general, nonsimple, parameterization see Berg (1992).

The notation is uncluttered by dropping, for the most part, the explicit references to the partition indices. Thus, any reference to a pencil or matrix in this section should be taken as applying only to the partition under discussion. The following illustrates the subscript notation for referring to partitions of the deformation:

$$\boldsymbol{M}(s) = \begin{bmatrix} \boldsymbol{M}_z & & & & \\ & \boldsymbol{M}_\epsilon & & & \\ & & \boldsymbol{M}_\eta & & \\ & & & \boldsymbol{M}_\infty & \\ & & & & \boldsymbol{M}_f \end{bmatrix} \Rightarrow \boldsymbol{\Delta}(s)$$

$$= \begin{bmatrix} \boldsymbol{\Delta}_{zz} & \boldsymbol{\Delta}_{z\epsilon} & \boldsymbol{\Delta}_{z\eta} & \boldsymbol{\Delta}_{z\infty} & \boldsymbol{\Delta}_{zf} \\ \boldsymbol{\Delta}_{\epsilon z} & \boldsymbol{\Delta}_{\epsilon\epsilon} & \boldsymbol{\Delta}_{\epsilon\eta} & \boldsymbol{\Delta}_{\epsilon\infty} & \boldsymbol{\Delta}_{\epsilon f} \\ \boldsymbol{\Delta}_{\eta z} & \boldsymbol{\Delta}_{\eta\epsilon} & \boldsymbol{\Delta}_{\eta\eta} & \boldsymbol{\Delta}_{\eta\infty} & \boldsymbol{\Delta}_{\eta f} \\ \boldsymbol{\Delta}_{\infty z} & \boldsymbol{\Delta}_{\infty\epsilon} & \boldsymbol{\Delta}_{\eta} & \boldsymbol{\Delta}_{\infty\infty} & \boldsymbol{\Delta}_{\infty f} \\ \boldsymbol{\Delta}_{fz} & \boldsymbol{\Delta}_{f\epsilon} & \boldsymbol{\Delta}_{f\eta} & \boldsymbol{\Delta}_{f\infty} & \boldsymbol{\Delta}_{ff} \end{bmatrix},$$

where the pencil $\boldsymbol{\Delta}(s)$ is the perturbation portion of the deformation, i.e.

$$\boldsymbol{\Delta}(s) = \sum_{n=1}^{k} c_n \boldsymbol{V}_n$$

and the deformation is $\boldsymbol{M}(s) + \boldsymbol{\Delta}(s)$. The details of the calculations for each partition are omitted, since they are extremely lengthy. The Appendix gives an example of the procedure for one class of partitions. For a full account, see Berg (1992).

6.1. *The partition* $\boldsymbol{\Delta}_{zz}$

The perturbation $\boldsymbol{\Delta}_{zz}$ is a $g \times h$ pencil, with $2gh$ free parameters:

$$\boldsymbol{\Delta}_1 = \begin{bmatrix} c_{111} & c_{121} & \cdots & c_{1h1} \\ c_{211} & c_{221} & \cdots & c_{2h1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{g11} & c_{g21} & \cdots & c_{gh1} \end{bmatrix}$$

$$\boldsymbol{\Delta}_2 = \begin{bmatrix} c_{112} & c_{122} & \cdots & c_{1h2} \\ c_{212} & c_{222} & \cdots & c_{2h2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{g12} & c_{g22} & \cdots & c_{gh2} \end{bmatrix}.$$

The deformation has gh causal free parameters and gh noncausal free parameters.

6.2. *The partitions* $\boldsymbol{\Delta}_{\epsilon\epsilon}$ *and* $\boldsymbol{\Delta}_{\eta\eta}$

The canonical perturbation for a typical partition of this type, say $\boldsymbol{\Delta}_{\epsilon_i\epsilon_j}$, is a $\varepsilon_i \times (\varepsilon_j + 1)$ pencil, with the following form:

If $\varepsilon_{j+1} \geq \varepsilon_i$, then $\boldsymbol{\Delta} = \boldsymbol{0}$
If $\varepsilon_j + 1 < \varepsilon_i$, then

$$\boldsymbol{\Delta}_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & & 0 & 0 \\ \vdots & & 0 & c_1 \\ \vdots & & 0 & c_2 \\ \vdots & & \vdots & \vdots \\ \vdots & & 0 & c_\delta \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Delta}_2 = 0,$$

where $\delta = \varepsilon_i - (\varepsilon_j + 1)$. The number of free parameters is δ. A common case is $\varepsilon_i = 1$. Then the above results show that $\boldsymbol{\Delta}_1$ is a row vector of zeros.

The canonical perturbation $\boldsymbol{\Delta}_{\eta_i\eta_j}$ is a $(\eta_i + 1) \times \eta_j$ pencil with the following form:

if $(\eta_i + 1) \geq \eta_j$, then $\boldsymbol{\Delta} = \boldsymbol{0}$;
if $(\eta_i + 1) < \eta_j$, then

$$\boldsymbol{\Delta}_1 = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & c_1 & c_2 & \cdots & c_\delta & 0 \end{bmatrix},$$

$$\boldsymbol{\Delta}_2 = 0,$$

where $\delta = \eta_j - (\eta_i + 1)$. The number of free parameters is δ. In the common case $\eta_j = 1$, then the perturbation is identically zero.

6.3. *The partitions* $\boldsymbol{\Delta}_{\infty\infty}$

The canonical perturbation for a typical partition of this type, say $\boldsymbol{\Delta}_{\rho_i\rho_j}$, is as follows:

For $\rho_i \geq \rho_j$,

$$\boldsymbol{\Delta}_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ c_{\rho_{j-1}} & \cdots & c_1 & 0 \end{bmatrix}, \quad \boldsymbol{\Delta}_2 = \begin{bmatrix} & \boldsymbol{0} \\ c_{\rho_j} & \end{bmatrix}.$$

For $\rho_i < \rho_j$,

$$\boldsymbol{\Delta}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{\rho_{i-1}} & 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{\Delta}_2 = \begin{bmatrix} & \boldsymbol{0} \\ c_{\rho_i} & \end{bmatrix}.$$

When $\rho = 1$ the special cases follow: (i) $\rho_i = \rho_j = 1$, $\boldsymbol{\Delta} = (0, c_1)$; (ii) $\rho_i = 1$, $\rho_j > 1$, $\boldsymbol{\Delta} = (0^{\mathrm{T}}, [c_1\ 0\ \ldots\ 0])$; (iii) $\rho_i > 1$, $\rho_j = 1$, $\boldsymbol{\Delta} = (0, [0\ \ldots\ 0\ c_1]^{\mathrm{T}})$.

6.4. *The partition* $\boldsymbol{\Delta}_{ff}$

The perturbation derived for this partition is causal, but not simple, in general. Under Restriction R the following result is obtained.

For the partitions corresponding to the distinct eigenvalues, $\boldsymbol{\Delta}_1$ is diagonal, with all parameters independent. For the zero eigenvalue, if it is not distinct, the matrix in Jordan form is partitioned according to its Jordan blocks. Then the perturbation of each of these partitions on and above the diagonal has the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ c_1 & c_2 & \cdots & c_k \end{bmatrix}.$$

The perturbation of each partition below the

diagonal has the form

$$\begin{bmatrix} c_1 & 0 & \dots & 0 \\ c_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_k & 0 & \dots & 0 \end{bmatrix}.$$

For all blocks, $\mathbf{\Delta}_2 = 0$.

Note that this is the the Jordan–Arnold form for $\mathbf{J}$ (Arnold, 1981, 1983; Gilmore, 1981). The total number of free parameters is

$$N_d + \sum_{k=1}^{N_z} (2k-1)n_k,$$

where N_d is the number of distinct, nonzero, eigenvalues of $\mathbf{J}$, N_z is the geometric multiplicity of the zero eigenvalue and n_k is the size of the kth Jordan block associated with the zero eigenvalue. The number of free parameters is correct even if Restriction R does not apply.

6.5. *The partitions* $\mathbf{\Delta}_{z\epsilon}$, $\mathbf{\Delta}_{\epsilon z}$, $\mathbf{\Delta}_{z\eta}$ *and* $\mathbf{\Delta}_{\eta z}$

The canonical perturbation $\mathbf{\Delta}_{z\epsilon}$ is a $g \times (\varepsilon + 1)$ pencil with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1(\varepsilon+1)} \\ c_{21} & c_{22} & \dots & c_{2(\varepsilon+1)} \\ \vdots & \vdots & & \vdots \\ c_{g1} & c_{g2} & \dots & c_{g(\varepsilon+1)} \end{bmatrix},$$

$$\mathbf{\Delta}_2 = \begin{bmatrix} 0 & \dots & 0 & c_{1(\varepsilon+2)} \\ 0 & \dots & 0 & c_{2(\varepsilon+2)} \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & c_{g(\varepsilon+2)} \end{bmatrix}.$$

Note that the matrix $\mathbf{\Delta}_2$ has $(\varepsilon + 1)$ columns. The parameters in the last column have subscript $(\varepsilon + 2)$ to distinguish them from the parameters in the last column of $\mathbf{\Delta}_1$. There are a total of $g(\varepsilon + 2)$ free parameters, with $g(\varepsilon + 1)$ of these in the causal part of the deformation.

The partitions of the canonical perturbation $\mathbf{\Delta}_{\epsilon z}$ are $\varepsilon \times h$ pencils with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1h} \\ \vdots & \vdots & & \vdots \\ c_{(\varepsilon-1)1} & c_{(\varepsilon-1)2} & \dots & c_{(\varepsilon-1)h} \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{\Delta}_2 = 0.$$

All $(\varepsilon - 1)h$ free parameters are associated with causal deformations.

The partitions of the canonical perturbation $\mathbf{\Delta}_{z\eta}$ are $g \times \eta$ pencils with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_{11} & \dots & c_{1(\eta-1)} & 0 \\ c_{21} & \dots & c_{2(\eta-1)} & 0 \\ \vdots & & \vdots & \vdots \\ c_{g1} & \dots & c_{g(\eta-1)} & 0 \end{bmatrix}, \quad \mathbf{\Delta}_2 = 0.$$

All $g(\eta - 1)$ free parameters are associated with causal deformations. If $\eta = 1$ then $\mathbf{\Delta}_1 = 0$.

Finally, the partitions of the canonical perturbation $\mathbf{\Delta}_{\eta z}$ are $(\eta + 1) \times h$ pencils with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1h} \\ c_{21} & c_{22} & \dots & c_{2h} \\ \vdots & \vdots & & \vdots \\ c_{(\eta+1)1} & c_{(\eta+1)2} & \dots & c_{(\eta+1)h} \end{bmatrix},$$

$$\mathbf{\Delta}_2 = \begin{bmatrix} 0 & 0 & \dots & c \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ c_{(\eta+2)1} & c_{(\eta+2)2} & \dots & c_{(\eta+2)h} \end{bmatrix}.$$

Note that the matrix $\mathbf{\Delta}_2$ has $(\eta + 1)$ rows. The parameters in the last row have subscript $(\eta + 2)$ to distinguish them from the parameters in the last row of $\mathbf{\Delta}_1$. There are a total of $h(\eta + 2)$ free parameters, $h(\eta + 1)$ of which are in the causal part of the deformation.

6.6. *The partitions* $\mathbf{\Delta}_{z\infty}$ *and* $\mathbf{\Delta}_{\infty z}$

The partitions of the canonical perturbation $\mathbf{\Delta}_{z\infty}$ are $g \times \rho$ pencils with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_{11} & \dots & c_{1(\rho-1)} & 0 \\ c_{21} & \dots & c_{2(\rho-1)} & 0 \\ \vdots & & \vdots & \vdots \\ c_{g1} & \dots & c_{g(\rho-1)} & 0 \end{bmatrix},$$

$$\mathbf{\Delta}_2 = \begin{bmatrix} c_{1\rho} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_{2\rho} & 0 & \dots & 0 \\ c_{g\rho} & 0 & \dots & 0 \end{bmatrix}.$$

There are a total of $g\rho$ free parameters, $g(\rho - 1)$ of which are associated with the causal deformation. If $\rho = 1$, than $\mathbf{\Delta}_1 = 0$ and $\mathbf{\Delta}_2$ is a column vector filled with independent parameters.

The partitions of the canonical perturbation $\mathbf{\Delta}_{\infty z}$ are $\rho \times h$ pencils with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & c_{2h} \\ \vdots & \vdots & & \vdots \\ c_{\rho 1} & c_{\rho 2} & \dots & c_{\rho h} \end{bmatrix},$$

$$\mathbf{\Delta}_2 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ c_{(\rho+1)1} & c_{(\rho+1)2} & \dots & c_{(\rho+1)h} \end{bmatrix}.$$

There are a total of ρh free parameters in the deformation, $(\rho - 1)h$ of which are associated with the causal portion of the deformation. If

$\rho = 1$, than $\mathbf{\Delta}_1 = 0$, and $\mathbf{\Delta}_2$ is a column vector of independent parameters.

6.7. *The partitions* $\mathbf{\Delta}_{\mathbf{zf}}$ *and* $\mathbf{\Delta}_{\mathbf{fz}}$

Typically, the $g \times n$ perturbation $\mathbf{\Delta}_{\mathbf{zf}}$ is not simple. However, under Restriction R it is. In that case each entry of $\mathbf{\Delta}_1$ is an independent parameter, and $\mathbf{\Delta}_2 = 0$. Thus, there is a total of ng free parameters in this partition, all causal. This total is valid whether or not Restriction R holds.

The perturbation $\mathbf{\Delta}_{\mathbf{fz}}$ is not simple either. Again, under Restriction R, the perturbation is an $n \times h$ pencil with each entry of $\mathbf{\Delta}_1$ an independent parameter, and $\mathbf{\Delta}_2 = 0$. The total number of free parameters is nh, all causal. Again, the total does not depend on Restriction R holding.

6.8. *The partitions* $\mathbf{\Delta}_{\epsilon\eta}$ *and* $\mathbf{\Delta}_{\eta\epsilon}$

The $\varepsilon \times \eta$ perturbation $\mathbf{\Delta}_{\epsilon\eta}$ is zero, $\mathbf{\Delta} = \mathbf{0}$. The $(\eta + 1) \times (\varepsilon + 1)$ canonical perturbation $\mathbf{\Delta}_{\eta\epsilon}$ has the form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_1 & c_2 & \dots & c_\eta & c_{\eta+1} \\ 0 & 0 & \dots & 0 & c_{\eta+2} \\ 0 & 0 & \dots & 0 & c_{\eta+3} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & c_{(\eta+\varepsilon+1)} \end{bmatrix},$$

$$\mathbf{\Delta}_2 = \begin{bmatrix} \mathbf{0} \\ c_{(\eta+\varepsilon+2)} \end{bmatrix}.$$

There are a total of $\eta + \varepsilon + 2$ free parameters, $\eta + \varepsilon + 1$ of which are associated with the causal portion.

6.9. *The partitions* $\mathbf{\Delta}_{\infty\epsilon}$, $\mathbf{\Delta}_{\epsilon\infty}$, $\mathbf{\Delta}_{\infty\eta}$ *and* $\mathbf{\Delta}_{\eta\infty}$

The partition $\mathbf{\Delta}_{\infty\epsilon}$ is a $\rho \times (\varepsilon + 1)$ pencil with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & c_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & c_{\rho-1} \end{bmatrix}, \quad \mathbf{\Delta}_2 = \begin{bmatrix} \mathbf{0} \\ c_\rho \end{bmatrix}.$$

There are a total of ρ free parameters, $\rho - 1$ of which are associated with the causal part. If $\rho = 1$ and $\mathbf{\Delta}_1 = 0$ and $\mathbf{\Delta}_2$ is a row vector of zeros, except the last element, which is a free parameter.

The perturbation $\mathbf{\Delta}_{\epsilon\infty}$ is a $\varepsilon \times \rho$ pencil, identically equal to zero, $\mathbf{\Delta}_{\epsilon\infty} = \mathbf{0}$. The perturbation $\mathbf{\Delta}_{\infty\eta}$ is a $\rho \times \eta$, identically zero, pencil, $\mathbf{\Delta}_{\infty\eta} = \mathbf{0}$.

The partition $\mathbf{\Delta}_{\eta\infty}$ is a $(\eta + 1) \times \rho$ pencil with the following form:

$$\mathbf{\Delta}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 \\ c_{\rho-1} & c_{\rho-2} & \dots & c_2 & c_1 & 0 \end{bmatrix},$$

$$\mathbf{\Delta}_2 = \begin{bmatrix} \mathbf{0} \\ c_\rho \end{bmatrix}.$$

There are a total of ρ free parameters, $\rho - 1$ of which are associated with the causal part. If $\rho = 1$ then $\mathbf{\Delta}_1 = 0$ and $\mathbf{\Delta}_2$ is a column vector of zeros, except for the last entry, which is a free parameter.

6.10. *The partitions* $\mathbf{\Delta}_{\mathbf{f}\epsilon}$, $\mathbf{\Delta}_{\epsilon\mathbf{f}}$, $\mathbf{\Delta}_{\mathbf{f}\eta}$ *and* $\mathbf{\Delta}_{\eta\mathbf{f}}$

The $n \times (\varepsilon + 1)$ pencil $\mathbf{\Delta}_{\mathbf{f}\epsilon}$ is simple under Restriction R. In that case

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ c_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_n & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{\Delta}_2 = 0.$$

There are n free parameters, all causal. This total is independent of Restriction R.

The perturbation $\mathbf{\Delta}_{\epsilon\mathbf{f}}$ is an $\varepsilon \times n$ pencil, identically zero, $\mathbf{\Delta}_{\epsilon\mathbf{f}} = \mathbf{0}$. The perturbation $\mathbf{\Delta}_{\mathbf{f}\eta}$ is an $n \times \eta$ pencil, identically zero, $\mathbf{\Delta}_{\mathbf{f}\eta} = \mathbf{0}$.

The $(\eta + 1) \times n$ pencil $\mathbf{\Delta}_{\eta\mathbf{f}}$ is simple under Restriction R. In that case

$$\mathbf{\Delta}_1 = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{\Delta}_2 = 0.$$

There is a total of n free parameters, all causal, regardless of Restriction R.

6.11. *The partitions* $\mathbf{\Delta}_{\infty\mathbf{f}}$ *and* $\mathbf{\Delta}_{\mathbf{f}\infty}$

$\mathbf{\Delta}_{\infty\mathbf{f}} = \mathbf{0}$ and $\mathbf{\Delta}_{\mathbf{f}\infty} = \mathbf{0}$.

7. EXAMPLES

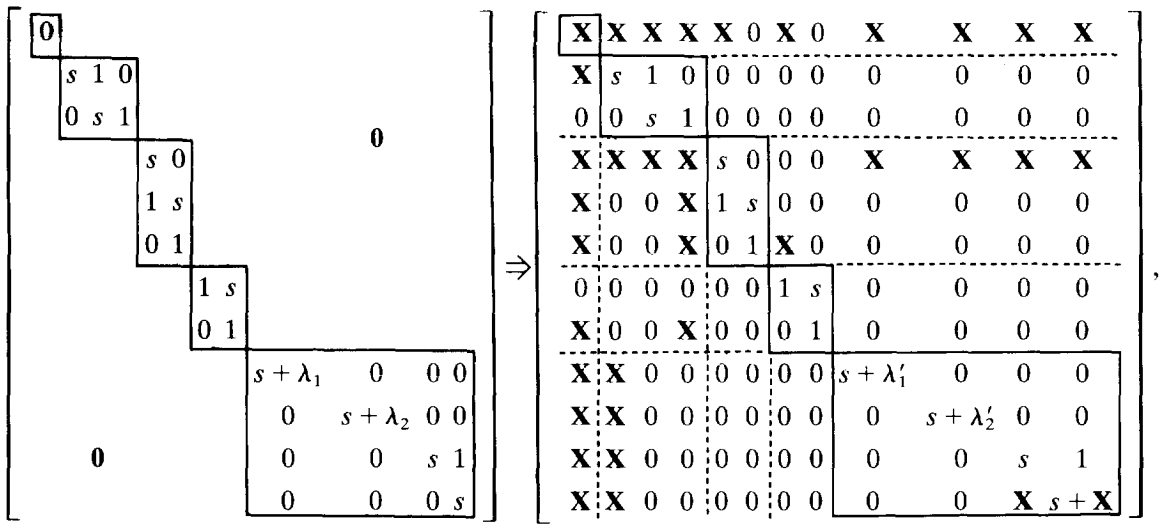
7.1. *Example* 1

The pencil has Kronecker indices $\varepsilon_1 = 0$, $\varepsilon_2 = 2$, $\eta_1 = 0$, $\eta_2 = 2$, $\rho = 2$, and

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The resulting pencil, and corresponding defor-

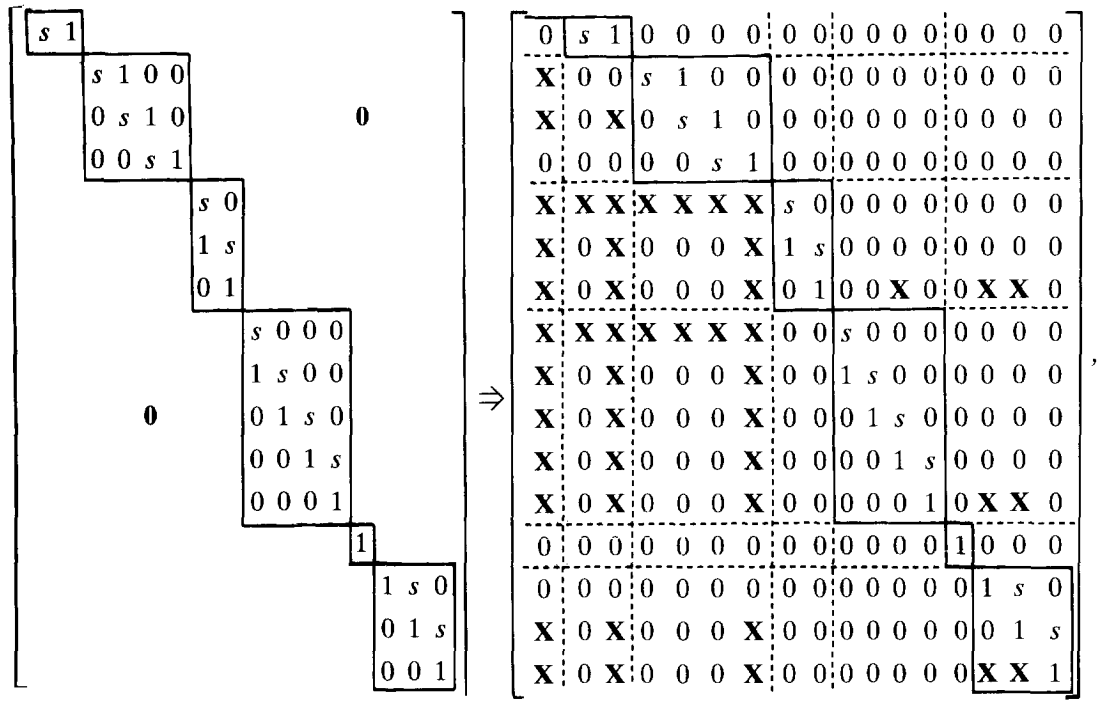
mation, are



where each **X** represents an independent parameter, and the primes indicate a perturbed value.

7.2. *Example* 2

The pencil has Kronecker indices $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $\varepsilon_3 = 3$, $\eta_1 = 2$, $\eta_2 = 4$, $\rho_1 = 1$ and $\rho_2 = 3$:



where again each **X** represents an independent parameter.

8. CONCLUSIONS

As the values of the parameters in a parameterized pencil are varied, the structural invariants of the pencil may change. Points at which this occurs are called structurally unstable and can be examined using the framework of singularity theory. The usual procedure is to derive a miniversal parameterization of the unstable structure (singularity), and then to describe all possible neighboring structures using the miniversal parameterization as a guide. Together, these steps are an 'unfolding' of the

singularity (Golubitsky and Schaeffer, 1984). Some control strategies require a particular zero structure, such as adaptive controllers, where the relative degree must be known, and robust regulators, where the system zeros are restricted. When these requirements break down, unfolding the singular point may help understand causes and solutions. Some preliminary applications along these lines are presented by Berg and Kwatny (1993).

This paper derives a miniversal parameterization of the Kronecker form for matrix pencils. The parameterized canonical form presented here comes with three caveats. The first is that it is valid only in some neighborhood of the nominal pencil. At this time there are no guidelines for determining the size of that neighborhood. The second is that the parameters take complex values, and some pencils so obtained may not correspond to physically realizable systems. This means some care is required in application. The third is that the nonzero finite elementary divisors must be distinct. This restriction can be relaxed, but the parameterization becomes cluttered. Berg (1992) provides details.

The number of parameters in the canonical deformation is of interst by itself. It provides some indication of the 'likelihood' of a particular singular case occuring in practice. The more parameters required to make a singularity generic in a family of pencils, the less likely that singularity is in a physical system with uncertain parameters. That number can be found from the preceding results by summing the values supplied for each partition. This calculation has also been made, independently, by Demmeland Edelman (1993).

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APPENDIX: SAMPLE CALCULATION USING A PARTITION OF $\Delta_{\epsilon\epsilon}$

Denote the partition by $\Delta_{\epsilon_i\epsilon_j}$. The relevant partitions of the nominal pencil are $[\![\mathbf{M}_1]\!]_{ii} = [0\ \mathbf{I}^{(\epsilon_i)}]$, $[\![\mathbf{M}_2]\!]_{ii} = [\mathbf{I}^{(\epsilon_i)}\ 0]$, $[\![\mathbf{M}_1]\!]_{jj} = [0\ \mathbf{I}^{(\epsilon_j)}]$, $[\![\mathbf{M}_2]\!]_{jj} = [\mathbf{I}^{(\epsilon_j)}\ 0]$. Write $\mathbf{P}$ and $\mathbf{Q}$ for $[\![\mathbf{U}_1^\dagger]\!]_{ji}$ and $[\![\mathbf{U}_2^\dagger]\!]_{ji}$. $\mathbf{P}$ and $\mathbf{Q}$ have dimension $(\varepsilon_j+1)\times\varepsilon_i$. The orthogonality conditions (14a,b) give the two matrix equations $[0\ \mathbf{I}^{(\epsilon_j)}]\mathbf{P} + [\mathbf{I}^{(\epsilon_j)}\ 0]\mathbf{Q} = 0$ and $\mathbf{P}[0\ \mathbf{I}^{(\epsilon_i)}] + \mathbf{Q}[^{(\epsilon_i)}\ 0] = 0$.

The first equation gives $[\mathbf{P}]_{(k+1)l} = -[\mathbf{Q}]_{kl}$ for $k = 1, \ldots, \varepsilon_j$, $l = 1, \ldots, \varepsilon_i$. The second equation gives $[\mathbf{P}]_{kl} = -[\mathbf{Q}]_{k(l+1)}$ for $k = 1, \ldots, \varepsilon_j + 1$, $l = 1, \ldots, \varepsilon_i - 1$, with $[\mathbf{P}]_{k\varepsilon_i} = 0$ for $k = 1, \ldots, \varepsilon_j + 1$ and $[\mathbf{Q}]_{k1} = 0$ for $k = 1, \ldots, \varepsilon_j + 1$. Combine these results to get

$$[\mathbf{P}]_{kl} = [\mathbf{P}]_{(k+1)(l+1)} \quad \text{for } k = 1, \ldots, \varepsilon_j,\ l = 1, \ldots, \varepsilon_i - 1,$$
$$[\mathbf{P}]_{kl} = 0 \quad \text{for } k = 2, \ldots, \varepsilon_j + 1.$$

So (i) the elements of $\mathbf{P}$ are constant along the diagonals, i.e. $\mathbf{P}$ is Topelitz, (ii) the entire last column of $\mathbf{P}$ is zero, and (iii) the first column of $\mathbf{P}$ is zero except, possibly, for the first

element. Its easy to see that if $\varepsilon_j + 1 \geq \varepsilon_i$, i.e. if $\mathbf{P}$ is square, or 'thin', then $\mathbf{P} = 0$. The $\boldsymbol{L}_\varepsilon$ blocks themselves have one more column than they do rows, so the corresponding blocks on the diagonal of $\mathbf{P}$, which have one more row than columns, are zero. If $\mathbf{P} = 0$ then the orthogonality conditions require $\mathbf{Q} = 0$. Then $[\![\boldsymbol{U}]\!]_{ij} = \mathbf{0}$. This is the case on and above the partition diagonal.

If $\mathbf{P}$ is 'fat' then define $\delta \stackrel{\text{def}}{=} \varepsilon_i - (\varepsilon_j + 1)$. There will be δ free parameters in the parameterization of this partition. This is the case below the partition diagonal. Construct $\mathbf{P}$ by setting the first δ elements of the first row to be arbitrary, independent parameters. Set the remainder of the elements in the first row and column to be zero. Then the Toeplitz structure determines the rest of $\mathbf{P}$. The negative of $\mathbf{Q}$ is formed by shifting the columns of $\mathbf{P}$ one to the right, and setting the first column to zero. Take the conjugate transpose to get $[\![\boldsymbol{U}]\!]_{ij}$. Write $[\![\boldsymbol{U}]\!]_{ij}$ as $\boldsymbol{U}$:

$$\mathbf{U}_1 = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ \vdots & c_1 & \ddots & \vdots \\ c_\delta & \vdots & \ddots & 0 \\ 0 & c_\delta & & c_1 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & c_\delta \\ 0 & \dots & \dots & 0 \end{bmatrix}, \quad \mathbf{U}_2 = -\begin{bmatrix} 0 & \dots & \dots & 0 \\ c_1 & 0 & \dots & 0 \\ \vdots & C_1 & \ddots & \vdots \\ c_\delta & \vdots & \ddots & 0 \\ 0 & c_\delta & & c_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & c_\delta \end{bmatrix},$$

where $\delta = \varepsilon_i - (\varepsilon_j + 1)$. These expressions implicitly define a basis $\{\boldsymbol{U}_i\}$, spanning the δ-dimensional subspace of $(\mathbf{T}_{\boldsymbol{M}}\mathcal{O}_{\boldsymbol{M}})^\perp$ corresponding to this partition.

To simplify this structure, first separate the causal and noncausal terms of the perturbation. The mapping that generates vectors tangent to the orbit for this partition is $\mathbf{V}_1^{\mathcal{O}} = \mathbf{S}[0\ \mathbf{I}^{(\varepsilon_j)}] + [0\ \mathbf{I}^{(\varepsilon_i)}]\mathbf{T}$, $\mathbf{V}_2^{\mathcal{O}} = \mathbf{S}[\mathbf{I}^{(\varepsilon_j)}\ 0] + [\mathbf{I}^{(\varepsilon_i)}\ 0]\mathbf{T}$, where $\mathbf{S}$ has dimension $\varepsilon_i \times \varepsilon_j$, $\mathbf{T}$ has dimension $(\varepsilon_i + 1) \times (\varepsilon_j + 1)$, and are otherwise arbitrary.

Now consider a typical element of the basis $\boldsymbol{U}^\kappa \in \{\boldsymbol{U}_i\}$

$$\mathbf{U}_1^\kappa = \begin{matrix} \\ \uparrow \\ \kappa \\ \downarrow \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & \vdots \\ 1 & \ddots & \vdots \\ 0 & \ddots & 0 \\ \vdots & \ddots & 1 \\ \vdots & & 0 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{U}_2^\kappa = \begin{matrix} \\ \uparrow \\ \kappa + 1 \\ \downarrow \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} -\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & \vdots \\ 1 & \ddots & \vdots \\ 0 & \ddots & 0 \\ \vdots & \ddots & 1 \\ \vdots & & 0 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

Set $\mathbf{S} = 0$, $\mathbf{T} = [(\mathbf{U}_2^\kappa)^{\mathrm{T}}\ 0]^{\mathrm{T}}$. The resulting vector tangent to the orbit is, $\mathbf{V}_1^{\mathcal{O}} = \mathbf{U}_1^\kappa$ and $\mathbf{V}_2^{\mathcal{O}} = -\mathbf{U}_2^\kappa$, so that $\frac{1}{2}(\boldsymbol{V}^{\mathcal{O}} + \boldsymbol{U}^\kappa) = (\mathbf{U}_1^\kappa, 0)$. This accomplishes the goal of creating a causal perturbation.

Next set $[\mathbf{T}]_{(\kappa+n)n} = n$; for $n = 1, \dots, \varepsilon_j$ and all other elements of $\mathbf{T}$ to zero. Then define $\mathbf{S}$ by $[\mathbf{S}]_{kl} = [\mathbf{T}]_{kl}$ for $k = 1, \dots, \varepsilon_i$ and $l = 1, \dots, \varepsilon_j$. Adding the resulting vector tangent to the orbit to the perturbation eliminates every entry in the perturbation except the one in the last column. The deformation is now simple. This is the desired form.