

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 0.5t^{-1} & -0.6 \\ 0.7 & 0.2 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \quad (8)$$

where

$$u(t) = f(Y(t)), \quad f(X) = \|X\| \sin \frac{1}{\|X\|}. \quad (9)$$

Following the proof in Lemma 3, we have

$$D(t) = \begin{bmatrix} \exp(-2t) & \exp(-3t) \\ 0 & \exp(-3t) \end{bmatrix}.$$

Let $t_0 = 0$. In Theorem 3, $\|H\| \approx 1.08$, $\|D\| \leq \sqrt{3} \exp(-2t)$, $\|D^{-1}\| \leq \sqrt{3} \exp 3t$, $\|C\| \leq \sqrt{2}(t+5)^{-3} \exp -t$, $\|B\| \leq \sqrt{3} \exp 0.5t$, $K_f = 1$, we get $\Lambda \approx 0.449$, $G \approx 77.76$. Let $X = [X_1, X_2]^T$. So

$$\|X(t)\| \leq \frac{\sqrt{3} \sqrt{(X_1(0) + X_2(0))^2 + X_2(0)^2} \exp(-2t)}{1 - 1.567 \cdot (0.449 + \sqrt{3} G \sqrt{(X_1(0) + X_2(0))^2 + X_2(0)^2})}$$

for $\|X(0)\| < \delta$ and for some positive real number $\delta > 0$. Hence, the equilibrium at the origin for the system (7)-(9) is asymptotically stable.

IV. CONCLUSION

It is well-known that the stability of a linear system $\dot{X}(t) = A(t)X(t)$ depends on the eigenvalues of the matrix A . From Theorem 2, we see that the bilinear part, $B(t)X(t)u(t)$, is more important than the term $C(t)u(t)$ if $\|D^{-1}(t)C(t)\| < M$, $t \geq 0$ for some constant M and the linear part $A(t)X(t)$. Suppose the inverse matrix of D equals the transpose of D , i.e., $D^{-1} = D^T$ in Theorem 2. Then the boundedness of the solution $X(t)$ depends on the negative definiteness of the matrix $B(t)$. On the other hand, in Theorem 3, one can see that the terms, $A(t)X(t)$ and $C(t)u(t)$, almost dominate the bilinear term if Λ is very small where Λ does not depend on the bilinear part. It seems that the bilinear term can be ignored if the terms $A(t)X(t)$, $C(t)u(t)$ satisfy the hypotheses in Theorem 3. Therefore, the behavior of stability of a bilinear system is determined by all factors, $A(t)$, $B(t)$, $C(t)$ and the input function.

REFERENCES

[1] R. R. Mohler, *Bilinear Control Processes*. New York: Academic, 1973.
 [2] M. Espana and I. D. Landau, "Reduced order bilinear models for distillation columns," *Automatica*, vol. 14, pp. 345-355, 1978.
 [3] H. N. Koivo and R. Cojocariu, "An optimal control for flotation circuit," *Automatica*, vol. 13, pp. 37-45, 1977.
 [4] M. S. Mousa, R. K. Miller, and A. N. Michel, "Stability analysis of hybrid composite dynamical systems: Descriptions involving operators and differential equations," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 216-226, 1986.
 [5] —, "Stability analysis of hybrid composite dynamical systems: Descriptions involving operators and difference equations," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 603-615, 1986.
 [6] T. Ionescu and R. V. Monopoli, "On the stabilization of bilinear systems via hyperstability," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 280-284, 1975.
 [7] R. Longchamp, "Stable feedback control of bilinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 37-45, 1980.

[8] J. P. Quinn, "Stabilization of bilinear systems by quadratic feedback controls," *J. Math. Anal. Appl.*, vol. 75, pp. 66-80, 1980.
 [9] P. O. Gutman, "Stabilizing controllers for bilinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 917-922, 1981.
 [10] C. Gounaridis and N. Kalouptsidis, "Stability of discrete-time bilinear systems with constant inputs," *Int. J. Contr.*, vol. 43, pp. 663-669, 1986.
 [11] M. Slemrod, "Stabilization of bilinear control systems with applications to nonconservative problems in elasticity," *SIAM J. Contr. Optimiz.*, vol. 16, pp. 131-141, 1978.
 [12] X. Yang, L. K. Chen, R. M. Burton, "Stability of bilinear systems with output feedback," *Int. J. Contr.*, vol. 50, no. 5, pp. 2085-2092, 1989.
 [13] R. R. Mohler, C. Bruni, and A. Gandolfi, "A systems approach to immunology," *Proc. IEEE*, vol. 68, pp. 964-990, 1980.
 [14] R. R. Mohler, *Nonlinear Systems and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1989.
 [15] F. Brauer and J. Nohel, *Ordinary Differential Equations*, 2nd ed. Menlo Park, CA: W. A. Benjamin, 1973.

[16] E. F. Beckenbach and R. Bellman, *Inequalities*, 2nd rev. New York: Springer-Verlag, 1965.

Regulation of Relaxed Static Stability Aircraft

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Abstract—We formulate and solve a regulator problem for nonlinear parameter-dependent dynamics. It is shown that the problem is solvable except at parameter values associated with bifurcation of the equilibrium equations and that such bifurcations are inherently linked to the system zero dynamics. These results are applied to the study of the regulation of the longitudinal dynamics of aircraft.

I. INTRODUCTION

In order to achieve higher levels of maneuverability and efficiency, future aircraft will operate close to or even beyond open-loop stability boundaries. For example, reduction of horizontal tail size in order to achieve reduced fuel consumption results in loss of longitudinal static stability for sufficiently aft center of gravity locations [2], [16]. Fighter aircraft may operate at a high angle of attack or at a high roll rate where nonlinear effects cause loss of stability [5]. Such aircraft require augmentation by automatic flight control systems which induce the desired handling qualities over the full range of flight conditions. When operating near stability boundaries the system dynamics can be nonlinear in an essential way. Marginally stable dynamics can be dramatically sensitive to parametric changes because of nonlinear effects.

Some recent studies in flight mechanics characterize aircraft

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parametric loss of stability in terms of elementary local bifurcations, c.f. [12], [8], [13]. Such phenomenon, of course, are fundamentally nonlinear. These studies deal almost exclusively with open-loop dynamics under parameter variation and very little general theory is available concerning the design of feedback controls near bifurcation points. Performance can sometimes be dramatically improved by nonlinear feedback. In fact, substantial improvement is obtained by Garrard and Jordan [7] in recovery from stall by using a nonlinear feedback.

Abed and Fu [1] address the issue of stabilizability and the design of stabilizing state feedback controllers at bifurcation points. In this note, we consider the design of feedback regulators in which the two-fold goal includes stabilization as well as the elimination of errors in selected output variables. As do Abed and Fu, we focus on local design in which it is intended to achieve these objectives for all parameter values on a neighborhood of a nominal value. Our main objective is to establish conditions for the existence of solutions to the regulator problem and to examine their significance to the control of aircraft longitudinal dynamics. In Section II, we define the local regulator problem and develop necessary and (constructive) sufficient conditions for its solution. The regulator problem gives rise to a natural set of equilibrium equations which include the zero output error relations. In Section III, we establish the connection between the solvability of the regulator problem and the static bifurcation of these equations. Under reasonable assumptions on the plant, solutions to the regulator problem fail to exist only at bifurcation points. We give a characterization of local static bifurcation in terms of the open-loop plant zeros. In Section IV we apply these concepts to the analysis of the longitudinal dynamics of an aircraft. It is clearly shown how bifurcation points arise in these problems and why they affect solvability of the regulator problem. The relationships between bifurcation, system zeros, and dynamic and static stability are illustrated. Section V summarizes our main conclusions.

II. REGULATOR DESIGN

A. Definition of the Local Regulator Problem

Consider the nonlinear dynamical system

$$\dot{x} = f(x, u, \mu) \quad (2.1a)$$

$$y = g(x, \mu) \quad (2.1b)$$

$$z = h(x, \mu) \quad (2.1c)$$

where, $x \in R^n$ is the system state, $u \in R^m$ is the control, $y \in R^q$ is the measurement, and $z \in R^p$ is the regulated output, $\mu \in R^k$ is a parameter vector which may be composed of plant parameters, exogenous constant disturbances, and/or set points. We assume that the functions f , g , and h are smooth (sufficiently differentiable) and our objective is to design a feedback regulator which stabilizes a desired equilibrium point corresponding to $z = 0$. A triple (x^*, u^*, μ^*) is an *equilibrium point* of the open-loop dynamics if

$$F(x^*, u^*, \mu^*) = \begin{bmatrix} f(x^*, u^*, \mu^*) \\ h(x^*, \mu^*) \end{bmatrix} = 0. \quad (2.2)$$

We usually obtain equilibria by specifying μ^* and solving (2.2) for x^*, u^* . Then $y^* := g(x^*, \mu^*)$. Typically, it is expected that (2.2) will have solutions only if $m \geq p$. Since the number of controls can always be reduced, we assume henceforth that $m = p$.

In the following paragraphs we consider two types of regulators, either state feedback

$$u = k(x, \mu) \quad (2.3)$$

with $k(x^*, \mu^*) = u^*$, or dynamic feedback

$$u = \eta(v, y), \quad \dot{v} = \phi(v, y) \quad (2.4)$$

where $v \in R^r$ with $\phi(v^*, y^*) = 0$ and $\eta(v^*, y^*) = u^*$. Correspondingly, the closed-loop dynamics with state feedback are of the form

$$\dot{x}_L = f_L(x_L, \mu) := f(x_L, k(x_L, \mu), \mu), \quad x_L = x \in R^n \quad (2.5)$$

or in the case of dynamic feedback

$$\dot{x}_L = f_L(x_L, \mu) := \begin{bmatrix} f(x, \eta(v, g(x, \mu)), \mu) \\ \phi(v, g(x, \mu)) \end{bmatrix} \Big|_{\substack{x \\ v}}, \quad x_L = [x^t, v^t]^t \in R^{n+r}. \quad (2.6)$$

The regulator problem is defined as follows.

The local regulator problem: Determine a feedback control law of type (2.3) or (2.4) so that the following two conditions obtain:

1) *Stability:* For each $\mu \in U$, a neighborhood of μ^* , the closed-loop has an exponentially stable equilibrium point characterized by the function $\bar{x}_L(\mu)$ with $\bar{x}_L(\mu^*) = x^*$ and $k(x^*, \mu^*) = u^*$ in the case of state feedback, or $\bar{x}_L(\mu^*) = [x^{*t}, v^{*t}]^t$ and $\eta(v^*, y^*) = u^*$ in the case of dynamic feedback.

2) *Regulation:* $z(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $\mu \in U$, $x_L(0) \in X_L$, a neighborhood of $\bar{x}_L(\mu^*)$.

Remark 2.1: Consider the set of points in R^{n+m+k} that satisfy $F(x, u, \mu) = 0$ [(2.2)]

$$\mathcal{E} = \{(x, u, \mu) \in R^{n+m+k} | F(x, u, \mu) = 0\}. \quad (2.7)$$

We assume that $\text{rank}[D_x F D_u F D_\mu F] = n + m$ on \mathcal{E} . Then \mathcal{E} is a regular manifold of dimension k in R^{n+m+k} and we refer to \mathcal{E} as the open-loop equilibrium manifold. \mathcal{E} is the manifold of (output regulated) equilibria of the system (2.1). The nominal parameter value may be associated with several equilibria one of which is chosen as the equilibrium at which the regulator is to be designed. The actual parameter value will belong to a sufficiently small but unspecified neighborhood of μ^* . Our objective is to design a regulator which will identify and stabilize the corresponding equilibrium point.

The following proposition illustrates the intimate connection between stability and the existence of equilibria under parameter perturbations.

Theorem 2.1: Suppose x_L^* denotes an exponentially stable equilibrium point of the closed-loop dynamics at $\mu = \mu^*$. Then there exists a function $\bar{x}_L(\mu)$ on a neighborhood U of μ^* with $\bar{x}_L(\mu^*) = x_L^*$ and which satisfies the relation

$$f_L(\bar{x}_L(\mu), \mu) = 0. \quad (2.8)$$

Proof: By hypothesis we have an exponentially stable equilibrium point which satisfies $f_L(x_L^*, \mu^*) = 0$. Notice that exponential stability implies that the Jacobian $D_{x_L} f_L(x_L^*, \mu^*)$ is nonsingular. The result follows from application of the implicit function theorem. \blacklozenge

We employ the concepts of exponential stabilizability and detectability of (2.1) as characterized in [4]:

Definition 2.1: The system $\dot{x} = f(x, u)$, $x \in R^n$, $u \in R^m$ with $f(x^*, u^*) = 0$ is *exponentially stabilizable* at (x^*, u^*) if there exists a function $u = k(x)$, defined on a neighborhood of x^* in R^n with $u^* = k(x^*)$ so that the equilibrium point, $x = x^*$, of $\dot{x} = f(x, k(x))$ is exponentially stable.

Definition 2.2: The system $\dot{x} = f(x, u)$, $y = g(x)$, $x \in R^n$, $u \in R^m$, $y \in R^q$ with $f(x^*, u^*) = 0$ and $y^* := g(x^*)$ is *exponentially detectable* at (x^*, u^*) if there exists a system

$$\dot{\xi} = \gamma(\xi, y), \quad \xi \in R^r \quad (2.9)$$

where the function γ is defined on a neighborhood of (x^*, y^*) in $R^n \times R^q$ and the following conditions are satisfied:

$$\gamma(x^*, y^*) = 0, \quad \gamma(x, g(x)) = f(x, u^*) \quad (2.10)$$

and the point $\xi = x^*$ is an exponentially stable equilibrium of the system

$$\dot{\xi} = \gamma(\xi, y^*). \quad (2.11)$$

Remark 2.2: Exponential detectability implies that the system

$$\dot{v} = \gamma(v, y) + f(v, u) - f(v, u^*) \quad (2.12)$$

is a local observer in the sense that $\|x(t) - v(t)\| \rightarrow 0$ as $t \rightarrow \infty$ provided $x(t)$ remains sufficiently close to x^* .

Remark 2.3: If a system is exponentially stabilizable at (x^*, u^*) , the equilibrium point (x^*, u^*) can be stabilized by linear state feedback. If it is exponentially stabilizable and detectable at (x^*, u^*) , the equilibrium point (x^*, u^*) can be stabilized with linear dynamic feedback.

Let (x^*, u^*, μ^*) satisfy (2.2). We make the following standing assumptions:

A1: the number of regulated outputs is the same as the number of controls, $m = p$.

A2: The system $\dot{x} = f(x, u, \mu^*)$ is exponentially stabilizable at (x^*, u^*) .

A3: The composite system $\dot{x} = f(x, u, \mu)$, $\mu = 0$, $y = g(x, \mu)$ is exponentially detectable at (x^*, u^*, μ^*) .

B. Existence of Solutions and Construction of Regulators

We now provide the basic necessary and sufficient conditions for the existence of solutions to the local regulator problem and in doing so provide an explicit method for regulator design. Our development parallels well-known constructions for linear systems as described in [10] and its references, particularly [6]. Thus, we only sketch the proofs.

Theorem 2.2: Let

$$\dot{x}_L = f_L(x_L, \mu) \quad (2.13a)$$

$$z = h_L(x_L, \mu) \quad (2.13b)$$

denote a closed-loop system. Let (x_L^*, μ^*) be an equilibrium point satisfying

$$f_L(x_L^*, \mu^*) = 0 \quad (2.14a)$$

$$h_L(x_L^*, \mu^*) = 0 \quad (2.14b)$$

and suppose it is exponentially stable. Then the output z is regulated only if

$$\begin{bmatrix} D_\mu f_L(x_L^*, \mu^*) \\ D_\mu h_L(x_L^*, \mu^*) \end{bmatrix} \in \text{Im} \begin{bmatrix} D_{x_L} f_L(x_L^*, \mu^*) \\ D_{x_L} h_L(x_L^*, \mu^*) \end{bmatrix}. \quad (2.15)$$

Proof: As in Theorem 2.1, exponential stability implies the invertibility of $D_{x_L} f_L^*$ and the existence of a function $\bar{x}_L(\mu)$ which satisfies $f_L(\bar{x}_L(\mu), \mu) = 0$ with $\bar{x}_L(\mu^*) = x^*$ on a neighborhood U of μ^* . Regulation implies that $\bar{x}_L(\mu)$ also satisfies $h_L(\bar{x}_L(\mu), \mu) = 0$. Differentiation of these conditions leads to

$$-D_{x_L} h_L^* [D_{x_L} f_L^*]^{-1} D_\mu f_L^* + D_\mu h_L^* = 0$$

which is equivalent to (2.15). \blacklozenge

We are now in a position to establish necessary conditions for state feedback solution to the regulator problem. First, the perturbation equations associated with (2.1a) and (2.1c) at an equilibrium point (x^*, u^*, μ^*) may be written

$$\delta \dot{x} = A^* \delta x + E^* \delta \mu + B^* \delta u \quad (2.16a)$$

$$\delta z = C^* \delta x + F^* \delta \mu \quad (2.16b)$$

where

$$A^* := \frac{\partial f}{\partial x}(x^*, u^*, \mu^*), \quad B^* := \frac{\partial f}{\partial u}(x^*, u^*, \mu^*),$$

$$E^* := \frac{\partial f}{\partial \mu}(x^*, u^*, \mu^*) \quad (2.17a)$$

$$C := \frac{\partial h}{\partial x}(x^*, \mu^*), \quad F^* := \frac{\partial h}{\partial \mu}(x^*, \mu^*). \quad (2.17b)$$

Theorem 2.3: The local regulator problem at (x^*, u^*, μ^*) has a state feedback solution only if

$$\begin{bmatrix} E^* \\ F^* \end{bmatrix} \in \text{Im} \begin{bmatrix} A^* & B^* \\ C^* & 0 \end{bmatrix} \quad (2.18)$$

Proof: Let $u = k(x, \mu)$ with $k(x^*, \mu^*) = u^*$ be a solution, so that the closed-loop equations are

$$\dot{x} = f(x, k(x, \mu), \mu) \quad (2.19a)$$

$$z = h(x, \mu). \quad (2.19b)$$

Now the application of Theorem 2.2 to (2.19) leads to (2.18). \blacklozenge

Theorem 2.4: The local regulator problem at (x^*, u^*, μ^*) has a dynamic feedback solution only if it has a state feedback solution.

Proof: Application of Theorem 2.2 to the closed-loop equations (2.6) and (2.1c) and direct computation leads to (2.18). \blacklozenge

Now we give constructive sufficient conditions for regulator design.

Definition 2.3: An equilibrium point (x^*, u^*, μ^*) of (2.2) is *regular* if there exists a neighborhood of μ^* on which there exist unique continuously differentiable functions $\bar{x}(\mu)$, $\bar{u}(\mu)$ with $x^* = \bar{x}(\mu^*)$, $u^* = \bar{u}(\mu^*)$ satisfying

$$F(\bar{x}(\mu), \bar{u}(\mu), \mu) = 0. \quad (2.20)$$

Notice that the implicit function theorem implies that an equilibrium point is regular if

$$\det [D_x F D_\mu F]^* \neq 0. \quad (2.21)$$

Theorem 2.5: If the equilibrium point (x^*, u^*, μ^*) is regular, then

i) there exist functions $k_0: R^n \rightarrow R^m$, $\bar{x}: R^k \rightarrow R^n$, and $\bar{u}: R^k \rightarrow R^m$ so that the regulator problem has a state feedback solution in the form

$$u = k(x, \mu) := \bar{u}(\mu) + k_0(x - \bar{x}(\mu)) \quad (2.22)$$

ii) there exist functions k_0, \bar{x}, \bar{u} as in i) and functions $\gamma_1: R^n \times R^k \times R^q \rightarrow R^n$, $\gamma_2: R^k \times R^k \times R^q \rightarrow R^k$ so that the regulator problem has a dynamic feedback solution in the form

$$u = \bar{u}(v_2) + k_0(v_1 - \bar{x}(v_2)) = \eta(v_1, v_2) \quad (2.23a)$$

$$\dot{v}_1 = \gamma_1(v_1, v_2, y) + f(v_1, \eta(v_1, v_2), v_2)$$

$$-f(v_1, u^*, v_2), \quad \dot{v}_2 = \gamma_2(v_1, v_2, y). \quad (2.23b)$$

Proof:

i) We construct a state feedback compensator with the desired properties. Since the equilibrium point is regular, there exist functions $\bar{x}(\mu)$, $\bar{u}(\mu)$ which satisfy (2.20) and have the property $x^* = \bar{x}(\mu^*)$, $u^* = \bar{u}(\mu^*)$. Now, let $u = u^* + k_0(x - x^*)$ be a feedback controller with $k_0(0) = 0$ which exponentially stabilizes the equilibrium point (x^*, u^*, μ^*) and which exists by assumption. In fact $k_0(\xi) = -u^* + k(\xi + x^*)$, where $k(\cdot)$ is the stabilizing feedback of Definition 2.1. The closed-loop dynamics with controller (2.22)

are

$$\dot{x} = f(x, \bar{u}(\mu) + k_0(x - \bar{x}(\mu)), \mu) \quad (2.24)$$

and for each fixed μ (2.2) has a solution (\bar{x}, \bar{u}) . The perturbation equations are

$$\delta \dot{x} = A(\mu) \delta x, \quad A(\mu) = [D_x f + D_u f D_x k_0]_{x=\bar{x}}. \quad (2.25)$$

Since A is a continuous function of μ and exponentially stable at μ^* , it is exponentially stable on a neighborhood of μ^* . Regulation follows from the fact that (2.20) implies $h(\bar{x}(\mu), \mu) = 0$ for μ on a neighborhood of μ^* , so that $z = 0$ at equilibrium.

ii) Exponential observability of the composite system implies the existence of functions $\gamma_1(v_1, v_2, y)$ and $\gamma_2(v_1, v_2, y)$ with the properties

$$\gamma_1(x, \mu, g(x, \mu)) = f(x, u^*, \mu), \quad \gamma_2(x, \mu, g(x, \mu)) = 0 \quad (2.26)$$

and so that the dynamical system

$$\dot{v}_1 = \gamma_1(v_1, v_2, y^*), \quad \dot{v}_2 = \gamma_2(v_1, v_2, y^*) \quad (2.27)$$

has an exponentially stable equilibrium point $(v_1, v_2) = (x^*, \mu^*)$.

Now, the closed-loop dynamics are

$$\dot{x} = f(x, \eta(v_1, v_2), \mu) \quad (2.28a)$$

$$\dot{v}_1 = \gamma_1(v_1, v_2, g(x, \mu)) + f(v_1, \eta(v_1, v_2), v_2) - f(v_1, u^*, v_2) \quad (2.28b)$$

$$\dot{v}_2 = \gamma_2(v_1, v_2, g(x, \mu)). \quad (2.28c)$$

Let us define a state transformation $(x, v_1, v_2) \rightarrow (x, \epsilon_1, \epsilon_2)$ where

$$\epsilon_1 = v_1 - x, \quad \epsilon_2 = v_2 - \mu$$

so that the loop equations become

$$\dot{x} = f(x, \eta(x + \epsilon_1, \mu + \epsilon_2), \mu) \quad (2.29a)$$

$$\begin{aligned} \dot{\epsilon}_1 &= \gamma_1(x + \epsilon_1, \mu + \epsilon_2, g(x, \mu)) \\ &+ f(x + \epsilon_1, \eta(x + \epsilon_1, \mu + \epsilon_2), \mu + \epsilon_2) \\ &- f(x + \epsilon_1, u^*, \mu + \epsilon_2) - f(x, \eta(x + \epsilon_1, \mu + \epsilon_2), \mu) \end{aligned} \quad (2.29b)$$

$$\dot{\epsilon}_2 = \gamma_2(x + \epsilon_1, \mu + \epsilon_2, g(x, \mu)). \quad (2.29c)$$

It is easy to verify that an equilibrium point of (2.29) is $(x, \epsilon_1, \epsilon_2) = (\bar{x}(\mu), 0, 0)$ by making use of (2.26). Now some lengthy calculations show that the perturbation equations associated with (2.29) are of the form

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \end{bmatrix} &= A(\mu) \begin{bmatrix} x \\ \epsilon_1 \\ \epsilon_2 \end{bmatrix}, \quad \text{with } A(\mu) \\ &= \begin{bmatrix} [(D_x f) + (D_u f)(D_x k_0)]^* & ** & ** \\ 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix} \\ &+ O(\mu - \mu^*) \end{aligned} \quad (2.30)$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} D_{v_1} \gamma_1(x^*, \mu^*, y^*) & D_{v_2} \gamma_1(x^*, \mu^*, y^*) \\ D_{v_1} \gamma_2(x^*, \mu^*, y^*) & D_{v_2} \gamma_2(x^*, \mu^*, y^*) \end{bmatrix}.$$

Since $k_0(x)$ is an exponentially stabilizing state feedback controller, it follows that $[(D_x f) + (D_u f)(D_x k_0)]^*$ has all of its eigenvalues in the open left-half plane. Similarly, exponential observability

implies that Γ (the perturbation matrix associated with (2.27)) has all of its eigenvalues in the open left-half plane and this property is therefore inherited by $A(\mu^*)$. It follows from continuity of the eigenvalues of $A(\mu)$ that there is a neighborhood of μ^* on which $A(\mu)$ is exponentially stable. \blacklozenge

C. Linear Solution of the Local Regulator Problem

Remark 2.2 suggests solvability by linear feedback. It will be assumed that

$$1) \quad \text{Im} \begin{bmatrix} A^* & B^* \\ C^* & 0 \end{bmatrix} = \mathbb{R}^{n+p} \quad (2.31)$$

2) z is readable from y , i.e., there exists a matrix Q such that $z = Qy$.

We refer to 1) as the *strong regularity condition*. Now, we design a (constant) disturbance accommodating regulator for the surrogate perturbation system

$$\delta \dot{x} = A^* \delta x + G\omega + B^* \delta u \quad (2.32a)$$

$$\dot{\omega} = 0 \quad (2.32b)$$

$$\delta z = C^* \delta x + H\omega \quad (2.32c)$$

where $\omega \in \mathbb{R}^p$ (recall $p = \dim(z)$) and G, H are chosen (as they always may be) so that the composite state is detectable in δz . Such a compensator is constructed as follows.

i) Determine matrices X, U , respectively, $n \times p, m \times p$, which satisfy the matrix equations

$$A^* X + B^* U + G = 0 \quad (2.33a)$$

$$C^* X + H = 0. \quad (2.33b)$$

ii) Determine an $m \times n$ matrix K_0 such that $(A^* + B^* K_0)$ is stable.

iii) Determine an $(n+p) \times p$ matrix $L = [L_1^T L_2^T]$ so that the following matrix is stable:

$$\left\{ \begin{bmatrix} A^* & G \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [C^* \quad H] \right\}.$$

The required compensator is then

$$u = u^* + Uv_2 + K_0(v_1 - Xv_2) \quad (2.34a)$$

$$\begin{aligned} \dot{v}_1 &= (A^* + L_1 C^* + B^* K_0)v_1 \\ &+ (G + L_1 H + B^*(U - K_0 X))v_2 - L_1 Q(y - y^*) \end{aligned} \quad (2.34b)$$

$$\dot{v}_2 = (L_2 C^*)v_1 + (L_2 H)v_2 - L_2 Q(y - y^*). \quad (2.34c)$$

Lemma 2.1: Suppose that the equilibrium point (x^*, u^*, μ^*) satisfies the strong regularity condition (2.31). Then there exists matrices G, H such that the composite state of (2.32), $[\delta x^T, \omega^T]^T$ is detectable in the output δz and there exists matrices X, U satisfying (2.33). Moreover, for such G and H , $\text{Im}[U] = \mathbb{R}^m$.

Proof: Exponential detectability implies that (C^*, A^*) is a detectable pair. In an even more general context Francis [6] proves that when this is the case, $H = 0$ and almost every G renders the composite system detectable. The strong regularity condition insures that (2.33) is solvable for X, U for every G, H . It remains to show that $\text{Im}[U] = \mathbb{R}^m$. Assume that this is not the case. Then there exists a vector v having the property that

$$v \in \text{Im} \begin{bmatrix} A^* \\ C^* \end{bmatrix} \quad \text{and} \quad v \in \text{Im} \begin{bmatrix} G \\ H \end{bmatrix}. \quad (2.35)$$

Now recall that that detectability of the composite system implies

that

$$\ker \left\{ \begin{bmatrix} C^* & H \end{bmatrix} \right\} \cap \ker \left\{ \begin{bmatrix} A^* & G \\ 0 & 0 \end{bmatrix} - \lambda I_{n+p} \right\} = 0$$

for all λ in the closed right-half plane. (2.36)

But in view of (2.35), the matrix

$$\begin{bmatrix} A^* & G \\ C^* & H \end{bmatrix}$$

is singular so that (2.36) fails with $\lambda = 0$. Hence, we have a contradiction. \blacklozenge

Now, we can establish the following result.

Theorem 2.6: If the equilibrium point (x^*, u^*, μ^*) satisfies the strong regularity condition and if z is readable from y with $z = Qy$, then there exist matrices X, U, K_0, L as required for the linear compensator (2.34) and this compensator is a solution of the local regulator problem.

Proof: First, note that the assumptions of exponential stabilizability and detectability assure the existence of the compensator parameters K_0, L . The existence of X, U is established by Lemma 2.1. Now we must show that the closed-loop has an equilibrium point corresponding to $z = 0$ for each μ on a neighborhood of μ^* , and that this equilibrium point is exponentially stable. Let us write the loop equations in the form

$$\dot{x} = f(x, u, \mu) \quad (2.37a)$$

$$z = h(x, \mu) \quad (2.37b)$$

$$u = u^* + Uv_2 + K_0(v_1 - Xv_2) \quad (2.37c)$$

$$\dot{v}_1 = (A^* + L_1C^*)v_1 + (G + L_1H)v_2 + B^*(u - u^*) - L_1z \quad (2.37d)$$

$$\dot{v}_2 = (L_2C^*)v_1 + (L_2H)v_2 - L_2z. \quad (2.37e)$$

First, we argue that there exist functions on $\bar{x}(\mu), \bar{u}(\mu), \bar{v}_1(\mu), \bar{v}_2(\mu)$ a neighborhood of μ^* with $\bar{x}(\mu^*) = x^*, \bar{u}(\mu^*) = u^*, \bar{v}_1(\mu^*) = 0, \bar{v}_2(\mu^*) = 0$ which satisfy the equilibrium equations

$$0 = f(\bar{x}, \bar{u}, \mu) \quad (2.38a)$$

$$0 = h(\bar{x}, \mu) \quad (2.38b)$$

$$\bar{u} = u^* + U\bar{v}_2 + K_0(\bar{v}_1 - X\bar{v}_2) \quad (2.38c)$$

$$0 = (A^* + L_1C^*)\bar{v}_1 + (G + L_1H)\bar{v}_2 + B^*(\bar{u} - u^*) \quad (2.38d)$$

$$0 = (L_2C^*)\bar{v}_1 + (L_2H)\bar{v}_2. \quad (2.38e)$$

Existence of functions $\bar{x}(\mu), \bar{u}(\mu)$ satisfying (2.38a) and (2.38b) is guaranteed by the assumption that (x^*, u^*, μ^*) is a regular equilibrium point. Since, by Lemma 2.1, $\text{Im}[U] = R^m$, there exists $\bar{v}_2(\mu)$ such that $\bar{u}(\mu) - u^* = U\bar{v}_2(\mu)$. Thus, (2.38c)–(2.38e) are satisfied with $\bar{v}_1(\mu) = X\bar{v}_2(\mu)$. It remains only to show that this equilibrium point is exponentially stable. We omit these calculations which proceed as in the proof of Theorem 2.5. \blacklozenge

Remark 2.4: An early application of this type of linear regulator to a nonlinear plant was in the control of electric power plants [11]. The nonlinear and linear regulators achieve output regulation via different mechanisms. In the former case, the closed-loop equilibrium point is defined by the functions $\bar{x}(\mu), \bar{u}(\mu)$, whereas, in the linear case “integral action” provides regulation.

III. STABILITY, BIFURCATION, AND ZERO DYNAMICS

A. Bifurcation Points

It is to be anticipated that feedback regulation at operating conditions with multiple equilibria which are in close proximity may

prove troublesome. We have already seen, in Theorem 2.1, that a necessary condition for exponential stability of an equilibrium point of a closed-loop system at a parameter value μ^* is the existence of an isolated equilibrium for each μ in a neighborhood of μ^* . We now develop a deeper perspective of the significance of multiple equilibria to regulator design.

Definition 3.1: An equilibrium point (x^*, u^*, μ^*) of (2.2) is a (static) bifurcation point with respect to $F(x, u, \mu)$ if in each neighborhood of (x^*, u^*, μ^*) there exists (x_1, u_1, μ) and (x_2, u_2, μ) with $(x_1, u_1) \neq (x_2, u_2)$ and $F(x_1, u_1, \mu) = 0, F(x_2, u_2, \mu) = 0$.

An equilibrium point is a bifurcation point if and only if it is not regular.

B. Bifurcation and Zero Dynamics

We can give a useful interpretation to static bifurcation for systems defined by state equations (2.1a) and output equations (2.1c). Equation (2.27) is equivalent to

$$\det \begin{bmatrix} A^* & B^* \\ C^* & 0 \end{bmatrix} \neq 0 \Leftrightarrow \text{Im} \begin{bmatrix} -A^* & B^* \\ -C^* & 0 \end{bmatrix} = R^{n+p}. \quad (3.1)$$

Thus, we have the following conclusion.

Theorem 3.1: An equilibrium point (x^*, u^*, μ^*) is a (static) bifurcation point only if

$$\text{Im} \begin{bmatrix} -A^* & B^* \\ -C^* & 0 \end{bmatrix} \neq R^{n+p}.$$

It follows that conditions for static bifurcation reduce to the following two possibilities.

1) If for typical λ

$$\text{rank} \begin{bmatrix} \lambda I - A^* & B^* \\ -C^* & 0 \end{bmatrix} = n + p \quad (3.2)$$

then a static bifurcation point corresponds to an invariant zero (of the linearized dynamics) at the origin. This is the *nondegenerate* case.

2) Otherwise, a static bifurcation point corresponds to the condition

$$\det \{ C^* [\lambda I - A^*]^{-1} B^* \} = \det \{ G(\lambda) \} = 0 \quad (3.3)$$

which implies insufficient independent controls or redundant regulated outputs. This is the *degenerate* case.

Remark 3.1: When it is possible to associate with (2.1a) and (2.1c) nonlinear zero dynamics in the sense of [3], then the nondegenerate case corresponds to a static bifurcation of the zero dynamics. The degenerate case corresponds to “structural instability” of the relative degree.

C. Simultaneous Regulation

It has been shown that the local regulator problem is solvable with a linear compensator at open-loop equilibria which satisfy the strong regularity condition. We give some insight into the limitations of a single linear compensator. In Remark 2.1, we defined the equilibrium manifold \mathcal{E} associated with the output regulated open-loop system (2.1). Now, consider the closed-loop system (2.37) which is the system (2.1) plus a linear regulator designed at a point $(x^*, u^*, \mu^*) \in \mathcal{E}$. The closed-loop equilibrium manifold is the set of points

$$\mathcal{E}_{cl} = \{ (x, v, u, \mu) \in R^{2n+2m+k} \mid \text{such that } (x, v, u, \mu) \text{ satisfies (2.38)} \}. \quad (3.4)$$

The structure of \mathcal{E}_{cl} is quite simple. It is to a k -dimensional

manifold whose projection on any R^{n+m+k} subspace of $R^{2n+2m+k}$ defined by $v = \text{constant}$ is precisely \mathcal{E} . There is a one-to-one correspondence between points in \mathcal{E} and \mathcal{E}_{cl} .

The bifurcation points themselves form codimension-1 manifolds in \mathcal{E} or \mathcal{E}_{cl} which divide them into open sets which we call *sheets*. The boundaries of these sets consist of the bifurcation points. Two different sheets are *contiguous* if they share common boundary points.

Theorem 3.2: A single linear compensator will generically fail to simultaneously solve the local regulator problem at two equilibria, one on each of two contiguous sheets of \mathcal{E} .

Proof: Let A and B denote two equilibria, one on each of two contiguous sheets of \mathcal{E} . There exist corresponding points A' , B' in \mathcal{E}_{cl} . We assume A' is a stable equilibrium of the closed-loop system and show that B' is generically unstable. Choose a path \mathcal{P} connecting A' and B' which transversely crosses a codimension one (in \mathcal{E}_{cl}) bifurcation surface at point C' . This is always possible because such paths are generic. Moreover, \mathcal{P} can be chosen so that C' is the only bifurcation point which it contains. C' is a codimension one (static) bifurcation point and generically corresponds to the coalescence of two hyperbolic closed-loop equilibria with the number of right-half plane eigenvalues differing by precisely one. C' itself corresponds to a closed-loop equilibrium point with precisely one zero eigenvalue. As the path \mathcal{P} is traversed from A' to B' , the only real eigenvalue crossing of the imaginary axis occurs at C' and this corresponds to a single eigenvalue. Hence, the number of right-half plane eigenvalues of A' and B' generically differ by an odd number. It follows that if A' is stable, B' is generically unstable. ♦

IV. AIRCRAFT LONGITUDINAL DYNAMICS

In this section, we illustrate the meaning and significance of the local regulator problem solvability conditions in the context of the control of aircraft longitudinal dynamics.

A. Equations of Motion

We summarize the basic equations of motion which govern the longitudinal dynamics of an aircraft. Further details may be found in [5]. Denote the velocity V , the body attitude θ , the flight path angle γ , the angle of attack $\alpha = \theta + \gamma$. The principle forces acting on the airframe are lift, drag, thrust, and weight. The body axis is aligned so that $\alpha = 0$ corresponds to zero lift. The basic equations of motion are

$$m(\dot{u} + w\dot{\theta}) = -mg \sin \theta + L_w \sin \alpha + L_t \sin \alpha_t + T - D \cos \alpha \quad (4.1a)$$

$$m(\dot{w} - u\dot{\theta}) = mg \cos \theta - L_w \cos \alpha - L_t \cos \alpha_t - D \sin \alpha \quad (4.1b)$$

$$I\ddot{\theta} = M_w + l_w L_w \cos \alpha - l_t L_t \cos \alpha_t - c\dot{\theta} \quad (4.1c)$$

where w , u are the velocity coordinates in the body frame

$$w = V \sin \alpha, \quad u = V \cos \alpha. \quad (4.2)$$

Also, α_t is the tail angle of attack and is related to the angle of attack α , pitch rate $\dot{\theta}$, tail angle i_t , downwash angle ϵ , and the elevator deflection angle δ via the relation

$$\alpha_t = \alpha + i_t - \epsilon + \delta + (l_t/V)\dot{\theta}. \quad (4.3)$$

We also have

$$l_w + l_t = 1^*. \quad (4.5)$$

The lift and drag forces depend on the velocity V , air density ρ , and

surface area S via the relations

$$L = C_L(\alpha) \frac{1}{2} \rho V^2 S, \quad D = C_D(\alpha) \frac{1}{2} \rho V^2 S, \quad M = C_M(\alpha) \frac{1}{2} \rho V^2 S. \quad (4.4)$$

Let us introduce a normalized velocity by identifying a nominal velocity (for example, the maximum cruise velocity) V_0 and define the nondimensional quantities

$$v := (V/V_0), \quad \kappa := (l_w/l_t^*), \quad \Lambda_w := (L_w/mg), \\ \Lambda_t := (L_t/mg), \quad \Delta := (D/mg), \\ \Pi := (T/mg), \quad \Sigma_w := (M_w/l_t^*mg), \\ \tau := (g/V_0)t, \quad q := (V_0/g)\dot{\theta}$$

in order to obtain the nondimensional equations

$$\begin{bmatrix} \cos \alpha & -v \sin \alpha & v \sin \alpha & 0 \\ \sin \alpha & v \cos \alpha & -v \cos \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{d}{d\tau} \begin{bmatrix} v \\ \alpha \\ \theta \\ q \end{bmatrix} = \begin{bmatrix} -\sin \theta + \Lambda_w \sin \alpha + \Lambda_t \sin \alpha_t + \Pi - \Delta \cos \alpha \\ \cos \theta - \Lambda_w \cos \alpha - \Lambda_t \cos \alpha_t - \Delta \sin \alpha \\ q \\ \frac{V_0^2 l_t^*}{gr^2} \{ \Sigma_w + \kappa \Lambda_w \cos \alpha - (1 - \kappa) \Lambda_t \cos \alpha_t \} - \frac{cV_0}{mgr^2} q \end{bmatrix}. \quad (4.6)$$

A complete set of aerodynamic properties must be defined for the purposes of numerical illustrative purposes. Although the model defined below does not correspond to any specific aircraft it does exhibit the general qualitative characteristics typically described in the literature. Level flight corresponds to $\gamma = 0$. We assume that the longitudinal body reference axis corresponds to the wing zero lift line and that level flight at nominal conditions (V_0, ρ_0) corresponds to $\alpha, \theta = \alpha_0$. In this case, the normalized lift forces take the form

$$A_w = f_w(\alpha) \bar{\rho} v^2, \quad \Lambda_t = f_t(\alpha_t) \bar{\rho} v^2, \\ \text{with } f_w(0) = 0, \quad f_t(0) = 0, \quad \bar{\rho} = (\rho/\rho_0). \quad (4.7)$$

The normalized drag force and moment are assumed to be of the form

$$\Delta = (a + b[f_w(\alpha)]^2) \bar{\rho} v^2, \quad \Sigma_w = \sigma_w(\alpha) \bar{\rho} v^2. \quad (4.8)$$

In the following discussion, numerical computations and examples will be based on the following model aircraft characteristics.

$$f_w = \left(\frac{\alpha - 2.08(\alpha - \alpha_0)^3}{\alpha_0} \right),$$

$$f_t = \epsilon' \left(\frac{(\alpha - \alpha_0 + \delta) - 3(\alpha - \alpha_0 + \delta)^3}{\alpha_0} \right),$$

$$\sigma_w(\alpha) = 0 \quad (4.9a)$$

$$\bar{\rho} = 1, \quad \epsilon = 0, \quad a = 0.05, \quad b = 0.05, \quad \alpha_0 = 0.05, \quad \epsilon' = 0.1,$$

$$\frac{V_0^2 l_t^*}{gr^2} = 300, \quad \frac{cV_0}{mgr^2} = 8. \quad (4.9b)$$

We note that $\kappa = 0$, $\delta = 0.0005$, $\Pi = 0.1$, $v = 1$, $\alpha = 0.0495$, $\theta = -0.0495$, $q = 0$ is an equilibrium point, corresponding to

level flight at nominal velocity. The perturbation equations are

$$\frac{d}{d\tau} \begin{bmatrix} \delta v \\ \delta \alpha \\ \delta \theta \\ \delta q \end{bmatrix} = \begin{bmatrix} -0.3960 & -2.949 & -1.0 & 0 \\ -1.980 & -21.80 & 0 & 1.0 \\ 0 & 0 & 0 & 1.0 \\ 0 & -599.2 & 0 & -8.0 \end{bmatrix} \begin{bmatrix} \delta v \\ \delta \alpha \\ \delta \theta \\ \delta q \end{bmatrix} + \begin{bmatrix} 0.9987 & 0.0010 \\ -0.0495 & -2.0 \\ 0 & 0 \\ 0 & -599.3 \end{bmatrix} \begin{bmatrix} \delta \Pi \\ \delta \end{bmatrix}.$$

B. Velocity Regulation

We give a simple example which illustrates the importance of center of gravity location on aircraft longitudinal static stability. With the elevator deflection angle δ fixed and the velocity v specified, we wish to determine values of α , θ (or, equivalently, γ) and Π which satisfy the equilibrium equations. Notice that $q = 0$ in equilibrium and that the first equation can always be satisfied by choosing

$$\Pi = \sin(\alpha - \gamma) - \Lambda_w \sin \alpha - \Lambda_t \sin(\alpha + \delta) + \Delta \cos \alpha. \quad (4.10)$$

Thus, we need only be concerned with the determination of α and θ from the remaining two equilibrium equations

$$\cos(\theta) - \Lambda_w \cos \alpha - \Lambda_t \cos(\alpha + \delta) - \Delta \sin \alpha = 0 \quad (4.11a)$$

$$\Sigma_w + \kappa \Lambda_w \cos \alpha - (1 - \kappa) \Lambda_t \cos(\alpha + \delta) = 0. \quad (4.11b)$$

Let us consider κ to be the only adjustable parameter. Since α and θ are the dependent variables we have

$$DF = [D_\alpha F D_\theta F D_\kappa F] = \begin{bmatrix} A & B & 0 \\ C & 0 & D \end{bmatrix} \quad (4.12)$$

$$A = \frac{\partial}{\partial \alpha} [-\Lambda_w \cos \alpha - \Lambda_t \cos(\alpha + \delta) - \Delta \sin \alpha],$$

$$B = -\sin \theta \quad (4.13a)$$

$$C = \frac{\partial}{\partial \alpha} [\Sigma_w + \kappa \Lambda_w \cos \alpha - (1 - \kappa) \Lambda_t \cos(\alpha + \delta)],$$

$$D = \Lambda_w \cos \alpha + \Lambda_t \cos(\alpha + \delta). \quad (4.13b)$$

Notice that $\det \{D_x f\} = -BC = 0$ only if either $\theta = n\pi$ for some integer n or $C = 0$. Thus, a static bifurcation occurs only if one of these conditions is satisfied simultaneously with (4.11). We can easily illustrate the significance of the case $C = 0$ as κ varies. Equation (4.11b) provides a relation between the center of gravity location (κ) and the angle of attack. There is a critical cg location κ_c (and an associated α_c, θ_c) which coincides with $C(\alpha_c, \theta_c, \kappa_c) = 0$. Note that C may be interpreted as the pitch stiffness. The equilibrium point is statically stable if $C > 0$, statically unstable if $C < 0$ and has neutral static stability if $C = 0$. In the preceding example, neutral static stability corresponds to a parameter value at which the equilibrium point is not regular—indeed, it corresponds to a bifurcation point. For $\kappa > \kappa_c$ there are no equilibrium solutions and for $\kappa < \kappa_c$ there are two. In the latter case, the equilibrium corresponding to $C < 0$ (the one on the left) is statically stable (but it may be dynamically unstable) whereas the other equilibrium corresponds to $C > 0$ and is certainly unstable.

It is interesting to note that the bifurcation associated with $C = 0$ is clearly apparent in the angle of attack curve. It is a simple matter

to compute the null space spanning vector of $DF = [D_\alpha F D_\theta F]$ for this case and observe that both the α and θ components are nonzero. On the other hand, the null space spanning vector associated with the other bifurcation condition ($\theta = n\pi$) is identically zero in the α component. This suggests that the pitch angle constitutes a better characterization of equilibrium behavior because both bifurcations should be evident in the pitch angle equilibrium curve. Indeed, this is evident in Fig. 1 where the complete equilibrium curves are illustrated with normalized velocity $v = 0.42$ and a elevator deflection angle $\delta = 0.03$. This curve should be compared with Fig. 2 which corresponds to $v = 1$.

Fig. 1 illustrates the qualitative characteristics of the linear perturbation model associated with different points on the equilibrium surface for the case of reduced speed. Beginning at the lower left and increasing the cg location parameter, note that stability is lost at $\kappa = 0.0770$ with a pair of complex conjugate poles moving into the right-half plane. Further increase causes these poles to become positive real, with one moving right and the other left. At the value $\kappa = 0.0773$ the leftmost of these poles reaches the origin simultaneously with a real zero. This pole continues moving left as the zero continues to move right as we follow the equilibrium curve which now corresponds to decreasing κ . Eventually, the zero reverses its direction and returns to the origin at $\kappa = 0.0728$, continuing to move left in the complex plane as the curve is followed with κ now increasing. Again the zero reverses its direction and returns to the origin at $\kappa = 0.0773$. A real left-half plane pole moves to the right reaching the origin simultaneously with the zero at $\kappa = 0.0773$. Neutral static stability at two of the bifurcation points is coincidental with the essential requirement of a zero at the origin.

With reference to Fig. 2, the bifurcation value of the parameter is $\kappa = 0.054$ which corresponds to $\theta = 0$. At all points on the equilibrium surface the linear perturbation model with control input Π and regulated output v has three zeros—a complex conjugate pair and one real. The conjugate pair changes very little from point to point. Their locations are $-15.149 \pm j13.661$ at bifurcation. As the equilibrium curve is traversed counterclockwise the real zero moves from the left-half plane to the right, passing through zero at the bifurcation point. Following the same path, we note that the perturbation system is stable at all point until the point $\kappa = 0.051$, $\theta = 0.4$ is reached. At this point a pair of complex conjugate roots (the phugoid pair) cross the imaginary axis into the right-half plane.

Since the angle of attack is quite small and positive, the lower half of the curve corresponds to descent and essentially all of the upper corresponds to climb. Recall that the thrust Π varies along the equilibrium curve. For example, with $\kappa = 0$, Fig. 2 indicates two equilibria—an unstable climb with positive pitch attitude and a stable descent with a negative pitch attitude. The descent corresponds to a lower thrust than the climb.

C. Flight Path Regulation

A somewhat more pertinent example with respect to control system design is the following. Once again consider the longitudinal dynamics defined by (4.5). It is desired to regulate the velocity and flight path angle v^*, γ^* by adjusting the elevator deflection angle and thrust δ, Π . Thus, we define the output equations

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} v - v^* \\ \gamma - \gamma^* \end{bmatrix} = \begin{bmatrix} v - v^* \\ \alpha - \theta - \gamma^* \end{bmatrix}. \quad (4.14)$$

Given the desired flight path parameters v, γ we wish to determine values of α, δ, Π which satisfy the equilibrium equations (2.2). Once again Π is directly determined and we need only be concerned

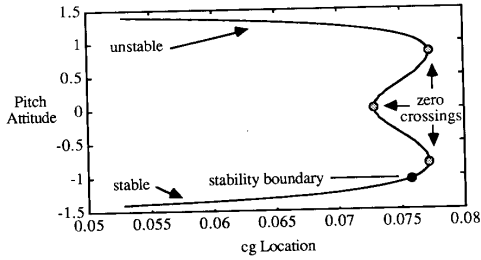


Fig. 1. Pitch attitude θ versus cg location κ with $v = 0.42$, and elevator deflection angle $\delta = 0.03$.

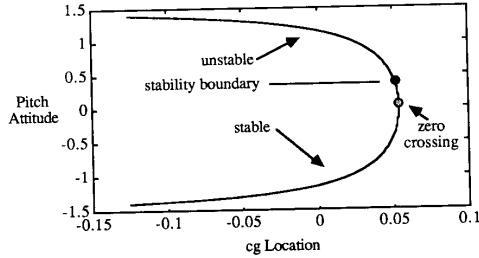


Fig. 2. Pitch attitude θ versus cg location κ with $v = 1.0$ and elevator deflection angle $\delta = 0.03$.

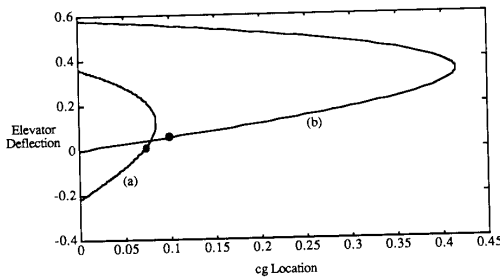


Fig. 3. Elevator deflection δ as a function of cg location κ . Curve (a) corresponds to level flight at reduced speed, $v = 0.42$, $\gamma = 0$ and (b) to cruise conditions $v = 1$, $\gamma = 0$. The dot denotes the stability boundary with the lower segment of the curves denoting open-loop stable equilibria.

with the solution properties of the pair of equations

$$\cos(\alpha - \gamma^*) - \Lambda_w \cos \alpha - \Lambda_l \cos(\alpha + \delta) - \Delta \sin \alpha = 0 \quad (4.15a)$$

$$\Sigma_w + \kappa \Lambda_w \cos \alpha - (1 - \kappa) \Lambda_l \cos(\alpha + \delta) = 0. \quad (4.15b)$$

Fig. 3 illustrates the equilibrium values of elevator deflection at cruise conditions and also at level flight with reduced velocity. We make the following comments. At cruise conditions stability is lost as the cg location moves aft with a single real root moving into the right-half plane. This occurs at approximately $\kappa = 0.089$ and corresponds to a loss of static stability. However, this does not correspond to a bifurcation point. Bifurcation occurs at $\kappa = 0.415$. The stability situation at low speed is more complicated. As κ increases from zero, a pair of complex conjugate (phugoid) roots move towards the imaginary axis and cross into the right-half plane at about $\kappa = 0.0710$. These roots meet on the positive real axis at approximately $\kappa = 0.0726$. One root then moves left the other right. The former reaches the origin at about $\kappa = 0.0727$, indicating neutral static stability. Of course the system is actually unstable.

Bifurcation occurs at $\kappa = 0.0838$. In both cases, the system has a pair of real zeros at all points of the equilibrium surface except the bifurcation point. These zeros move only slightly as the cg parameter changes. Just before and just after bifurcation these zeros have the locations: $-81.67, 73.67$ for cruise velocity and $-36.61, 28.67$ for reduced speed. The system has no (invariant) zeros at the bifurcation point. In fact, the perturbation system is degenerate at the bifurcation point which is readily observed from the state equations. Note that in both cases the two columns of the B matrix are linearly dependent.

Bifurcation Point, $v = 1$:

$$\frac{d}{d\tau} \begin{bmatrix} \delta v \\ \delta \alpha \\ \delta \theta \\ \delta q \end{bmatrix} = \begin{bmatrix} .0274 & -1.266 & -1.0 & 0 \\ -2.001 & -20.09 & 0 & 1.0 \\ 0 & 0 & 0 & 1.0 \\ 0 & 2481 & 0 & -8.0 \end{bmatrix} \begin{bmatrix} \delta v \\ \delta \alpha \\ \delta \theta \\ \delta q \end{bmatrix} + \begin{bmatrix} 0.9996 & 0.4749 \\ -0.0291 & -0.0138 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \Pi \\ \delta \delta \end{bmatrix}$$

Bifurcation Point, $v = 0.42$:

$$\frac{d}{d\tau} \begin{bmatrix} \delta v \\ \delta \alpha \\ \delta \theta \\ \delta q \end{bmatrix} = \begin{bmatrix} -1.866 & -1.566 & -1.0 & 0 \\ -4.286 & -3.070 & 0 & 1.0 \\ 0 & 0 & 0 & 1.0 \\ 0 & 59.43 & 0 & -8.0 \end{bmatrix} \begin{bmatrix} \delta v \\ \delta \alpha \\ \delta \theta \\ \delta q \end{bmatrix} + \begin{bmatrix} 0.9690 & 0.08121 \\ -0.2471 & -0.0207 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \Pi \\ \delta \delta \end{bmatrix}$$

Remark 4.3: Fig. 3 yields an interpretation of the power off stall [15]. The objective is to maintain level flight ($\gamma = 0$) while reducing speed to critical. Consider a family of constant speed, level flight path, equilibrium curves, two of which are illustrated in Fig. 3. Fix the cg location, for example at $\kappa = 0.8$. At nominal speed $v = 1$, curve (b) indicates an open-loop stable equilibrium. As speed is reduced, the equilibrium curve approaches curve (a). Eventually, the equilibrium point loses stability with a pair of complex eigenvalues migrating into the right-half plane suggesting a Hopf bifurcation. The pilot would experience the typical prestall buffeting. Further reduction of speed results in the vanishing of the equilibrium point, i.e., stall. Curve (a) illustrates the poststall situation. The perturbation system remains controllable (in the formal sense) even at the bifurcation point. Nevertheless, the actual aircraft becomes difficult to regulate as the bifurcation point (stall condition) is approached which is consistent with our necessary conditions that show that the aircraft cannot be regulated at such an equilibrium point.

V. CONCLUSIONS

In this note, we have investigated the design of feedback regulators for nonlinear parameter dependent dynamics. A concise formulation of the local regulator problem has been stated and solved. Under the assumptions of exponential stabilizability and detectability for the plant, the problem fails to be solvable only at static bifurcation points of the open-loop equilibrium equations. We have shown that static bifurcation is associated either with the presence of an invariant zero of the linearized (error) dynamics at the origin or with a degenerate transfer matrix. Our results have been developed in the spirit of the theory of linear disturbance accommodating regulators restricted to the constant disturbance case. Byrnes and Isidori [4] and also Jie and Rugh [9] consider nonlinear regulation with more general time-dependent but bounded disturbances. How-

ever, our focus on constant disturbances allows us to readily establish the connection with bifurcation theory. Local bifurcation behavior does help organize the global picture just as it does in dynamical systems theory, underscoring the significance of this association.

These results have been applied to the study of the regulation of longitudinal aircraft dynamics. It has been shown how center of gravity location affects the ability to regulate either velocity alone or velocity and flight path angle. In the former case, it was shown that the migration of a real zero through the origin is associated with a static bifurcation and we saw quite clearly why the regulator problem is not solvable—the equilibrium point vanishes under perturbation of the cg location. The latter case represents an example of a bifurcation associated with the degeneracy of the transfer matrix. In fact, in this case it is evident that at the bifurcation point the two controls are redundant thereby making it impossible to regulate two independent output variables. This is not an extraordinary situation. A simple computation shows that this is a generic possibility in one parameter families of models of the type employed in our analysis. Using linear models of the longitudinal dynamics of a fighter aircraft, Stengel [14] describes other control formulations in which the equivalent of our strong regularity condition fails. It is clear that the design engineer sets up the bifurcation behavior when the regulated outputs are selected. It is essential to consider the parametric dependence of the zero dynamics at an early stage of the control design process.

REFERENCES

- [1] E. H. Abed and J-H. Fu, "Local feedback stabilization and bifurcation control—Part I: Hopf bifurcation," *Syst. Contr. Lett.*, vol. 7, pp. 11–17, 1986; and "Local feedback stabilization and bifurcation control—Part II: Stationary bifurcation," *Syst. Contr. Lett.*, vol. 8, pp. 467–473, 1987.
- [2] J. D. Blight, D. Gangsaas, and T. M. Richardson, "Control law synthesis for an airplane with relaxed static stability," *J. Guidance, Contr. Dynamics*, vol. 9, no. 5, pp. 546–554, Sept.-Oct. 1986.
- [3] C. I. Byrnes and A. Isidori, "Local stabilization of minimum phase nonlinear systems," *Syst. Contr. Lett.*, vol. 10, pp. 9–17, 1988.
- [4] ———, "Output regulation of general nonlinear systems," (in French) to appear in *Automation (Theoretical)*.
- [5] B. Etkin, *Dynamics of Atmospheric Flight*. New York: Wiley, 1972.
- [6] B. A. Francis, "The linear multivariable regulator problem," *SIAM J. Contr.*, vol. 15, no. 5, pp. 486–505, 1977.
- [7] W. L. Garrard and J. M. Jordan, "Design of nonlinear automatic flight control systems," *Automatica*, pp. 497–505, 1977.
- [8] W. H. Hui and M. Tobak, "Bifurcation analysis of aircraft pitching motions about large mean angles of attack," *J. Guidance Contr.*, vol. 7, pp. 106–113, 1984.
- [9] H. Jie and W. J. Rugh, "On a nonlinear servomechanism problem," *Automatica*, to be published.
- [10] H. G. Kwatny and K. C. Kalnitsky, "On alternative methodologies for the design of robust linear multivariable regulators," *IEEE Trans. Automat. Contr.*, vol. AC-23, no. 5, pp. 330–339, Oct., 1978.
- [11] J. P. McDonald and H. G. Kwatny, "Design and analysis of boiler-turbine-generator controls using optimal linear regulator theory," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 202–209, 1973.
- [12] R. K. Mehra, W. C. Kessel, and J. V. Carrol, *Global Stability and Control Analysis of Aircraft at High Angles of Attack*. Cambridge, MA: Scientific Systems, 1977.
- [13] N. S. Namachchivaya and S. T. Ariaratnam, "Non-linear stability analysis of aircraft at high angles of attack," *Int. J. Non-Linear Mechanics*, vol. 21, no. 3, pp. 217–288, 1986.
- [14] R. F. Stengel, "Equilibrium response of flight control systems," *Automatica*, vol. 18, no. 3, pp. 343–348, 1982.
- [15] R. F. Stengel and W. B. Nixon, "Stalling characteristics of a general aviation aircraft," *J. Aircraft*, vol. 19, no. 6, pp. 425–434, 1982.
- [16] K. Wilhelm and D. Schafrank, "Landing approach handling qualities of transport aircraft with relaxed static stability," *J. Aircraft*, vol. 23, no. 10, pp. 756–762, 1986.

An Alternative Derivation of the Modified Gain Function of Song and Speyer

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Abstract—We show a much simpler derivation of the modified gain function for the bearings only measurement problem as defined by Song and Speyer [1]. The new form of the gain is more numerically stable than the original form. We also show the relationship between the modified gain and the standard gain of an extended Kalman filter. Finally, we confirm that the modified gain extended Kalman filter does indeed perform better than the standard extended Kalman filter.

INTRODUCTION

In [1], Song and Speyer define a modified gain extended Kalman filter (MGEKF) to handle a special class of nonlinear estimation problems. In particular, the nonlinearities must be "modifiable." The MGEKF was then applied to a tracker with bearings only measurements and shown to have improved performance over the standard extended Kalman filter (EKF). Here, we rederive the modified gains for the bearings only problem in a simpler manner. The form which results shows clearly the relationship between the modified gain and the standard gain. In the last section, we present some simulation results comparing the MGEKF to the EKF. We confirmed that the MGEKF does eliminate the erratic behavior exhibited by the EKF. Also, we found that the MGEKF filter state covariance estimate was a good predictor of filter performance.

DERIVATION

Song and Speyer define a modified gain extended Kalman filter (MGEKF) by the following set of equations:

$$\begin{aligned}\hat{X}_{i+1}^- &= \Phi_i \hat{X}_i^+ \\ \hat{X}_{i+1}^+ &= \hat{X}_{i+1}^- + k_{i+1}(z_{i+1} - h_{i+1}(\hat{X}_{i+1}^-)) \\ P_{i+1}^- &= \Phi_i P_i^+ \Phi_i^T + Q_i \\ k_{i+1} &= P_{i+1}^- \bar{h}_{i+1}^T (\bar{h}_{i+1} P_{i+1}^- \bar{h}_{i+1}^T + R)^{-1} \\ P_{i+1}^+ &= (I - k_{i+1} g(z_{i+1}, \hat{X}_{i+1}^-)) \\ &\quad \cdot P_{i+1}^- (I - k_{i+1} g(z_{i+1}, \hat{X}_{i+1}^-))^T + k_{i+1} R k_{i+1}^T\end{aligned}$$

where:

Φ_i	= Transition matrix at time i .
\hat{X}_i^-	= State estimate at time i before update.
\hat{X}_i^+	= State estimate at time i after update.
k_i	= Filter gain at time i .
z_i	= Measurement at time i .
$h_i(\hat{X}_i^-)$	= Predicted measurement based on state at time i before update.
P_i^-	= State covariance matrix at time i before update.
P_i^+	= State covariance matrix at time i after update.

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