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Variable structure control of systems with uncertain nonlinear friction☆

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Abstract

A new approach to control system design for systems containing sandwiched, uncertain, non-smooth friction is proposed. The method is based on a multi-state backstepping approach to variable structure control design. Stability and robustness properties are investigated and examples are given. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Many important control systems contain 'non-smooth' nonlinearities such as dead zone, backlash, hysteresis and coulomb friction that profoundly influence performance. While models exist for these effects, the parameters associated with them are almost always highly uncertain and can vary with time. Our main interest is precision pointing where friction is a dominant issue. Friction is typically nondifferentiable and uncertain, often of unknown functional form. Most high performance friction compensation methods are parameter adaptive systems that identify parameters of friction models of various degrees of complexity, e.g., (Armstrong-Helouvry, Dupont, & Canudas de Wit, 1994). When position-dependence or other effects encountered in practice are significant, the added model complexity can make this strategy unworkable. In this paper we seek robust controllers in which the non-smooth uncertainty is characterized simply in terms of a smooth bound. We impose no a priori structure on the friction function.

Although there is an extensive literature addressing robust control of systems with smooth uncertainties, non-smooth uncertainties have received comparatively little attention. Exceptions include the work of Tao and Kokotovic (1996) on the adaptive control of systems with non-smooth actuators and sensors and the growing literature on the adaptive control of non-smooth friction as noted above. To achieve 'ideal' performance nondifferentiable signals are generated to cancel or invert the nonlinearity. When the non-smooth nonlinearities are embedded in the dynamics of the system, it is not possible to produce the required nondifferentiable signals even when the nonlinearities are known with certainty. In this paper, we consider uncertain embedded or 'sandwiched' nonlinearities. We propose a control strategy based on sequential variable structure (VS) control design that generates approximately non-smooth cancelling signals.

If the system is smooth, input–output linearizable, and minimum phase, standard approaches to VS and adaptive control design require successive differentiation of the functions that define the system. Recently, Yip and Hedrick (1998) proposed a parameter adaptive control design strategy that avoids repeatedly differentiating the uncertainty. But they still require uncertainties with continuous first derivatives and the uncertainties need to be characterized as linear functions of uncertain parameters. Friction typically does not fit these criteria.

Specifically, we will consider single-input single-output systems (SISO) of the form:

$$\dot{x} = f(x) + \delta(x,t) + g(x)u,$$
$$y = h(x), \qquad (1)$$

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where the uncertainty $\delta(x,t)$ is piecewise continuous and the nominal system (f,g,h) is smooth and input–output linearizable.

2. A preliminary example

One approach to dealing with non-smooth plant nonlinearities is to approximate the non-smooth function by a smooth one. A naive application of that approach will almost certainly fail. Let us consider the following simple example that highlights the essential issues. Suppose we reduce the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\phi_{\rm fr}(x_2) + x_3, \quad \dot{x}_3 = u$$

to normal form. Let us write the friction model in the form of a nominal plus an uncertain part: $\phi_{\rm fr}(x_2) = \phi_{\rm fr0}(x_2) + \delta\phi_{\rm fr}(x_2)$, where the nominal part $\phi_{\rm fr0}(x_2)$ is smooth and the uncertainty $\delta\phi_{\rm fr}(x_2)$ is bounded. Then we have the coordinate transform

$$\begin{aligned} z_1 &= x_1 & & \dot{z}_1 = z_2, \\ z_2 &= x_2 & \Rightarrow \quad & \dot{z}_2 = z_3, \\ z_3 &= -\phi_{\rm fr}(x_2) + x_3, & & \dot{z}_3 = -\phi'_{\rm fr0}(z_2) - \delta\phi'_{\rm fr}(z_2) + u. \end{aligned}$$

Any error in the friction function produces an uncertainty that depends on the derivative $\delta\phi'_{\rm fr}(z_2)$. Obviously, if the friction function is nondifferentiable this will produce an unbounded, although matched, uncertainty. A feedback control cannot be made robust to this type of uncertainty. Let us instead base the normal form reduction on the smooth nominal system. Then we have the coordinate transform

$$\begin{aligned} z_1 &= x_1 & & \dot{z}_1 = z_2, \\ z_2 &= x_2 & \Rightarrow \quad & \dot{z}_2 = z_3 + \delta\phi_{\rm fr}(z_2), \\ z_3 &= -\phi_{\rm fr0}(x_2) + x_3 & & \dot{z}_3 = -\phi'_{\rm fr0}(z_2) + u. \end{aligned}$$

Now we have a bounded, although not matched, uncertainty.

Since it is generally not possible to reduce friction uncertainty to a bounded and matched form, we will use a backstepping approach. Backstepping was introduced in Kanellakapoulos, Kokotovic, and Morse (1991) for adaptive control design and adapted for recursive Lyapunov design in Freeman and Kokotovic (1993) for certain classes of systems having nonmatched uncertainty. Zinober and Liu (1996) have employed backstepping in conjunction with VS control to address smooth nonmatched uncertainty.

In the above example, the non-smooth, unmatched uncertainty is sandwiched between dynamical elements. A bounded and matched discontinuous nonlinearity could be cancelled (if known) by a bounded discontinuous control. In the application of backstepping to this example, we would want z_3 to act as a discontinuous pseudo-control that cancels $\delta\phi_{\rm fr}(z_2)$. This is of course not possible, unless u were allowed to be a singularity function. Our strategy will be to design a discontinuous VS control and then regularize it to achieve a smooth pseudo-control as required.

3. VS control with matched uncertainty

Our approach will require the design of smoothed VS controllers. We need to establish that our smoothed controller preserves the crucial property of the original (non-smooth) VS controller—robust stability with respect to matched, bounded uncertainty. Replacing an ideal switch with a smooth approximation in a variable structure controller does not always result in a stable system. We note one of several counter-examples in the literature.

Example 3.1. In Byrnes and Isidori (1989) it is shown that the origin of the system

$$\dot{x}_1 = x_2^2 + u, \quad \dot{x}_2 = x_1^2 - x_2^5, \quad y = x_1$$

cannot be stabilized by any smooth output feedback controller. On the other hand, it is easy to verify that the switching control $u = -\kappa \operatorname{sgn} y,\ \kappa > 0$ does asymptotically stabilize the origin. But any smooth approximation to the switch results in a smooth output feedback controller and hence must be unstable.

3.1. VS control design

There are two basic steps to designing a VS control: (1) design of the sliding control or equivalently the sliding surface, and (2) design of the reaching or switching control (see Utkin, 1978). Typically, a preliminary step reduces the system to normal, or regular, form. We will take as our starting point the system, already reduced to normal form:

$$\dot{\xi} = F(\xi, z), \tag{2}$$

$$\dot{z} = Az + b[\alpha(x(\xi,z)) + \Delta(\xi,z,t) + \rho(x(\xi,z))u], \tag{3}$$

where Δ is a bounded function that can represent uncertainties, disturbances and/or nondifferentiable functions. We assume

$$|\Delta(x(\xi,z),t)| < \sigma_\Delta(\xi,z) \quad \forall t,$$

where $\sigma_\Delta > 0$ is a continuous function. For the system (2), (3) with stable zero dynamics, we construct a variable structure control law with switching surface of the form, $s(x) = Kz(x)$, where K is chosen to stabilize the sliding mode dynamics (see Kwatny & Kim, 1990).

To insure that sliding occurs, we specify control functions $u^{\pm}(x)$ such that the manifold $s(x) = 0$ contains a stable submanifold. There are many ways of approaching the reaching design problem. One approach is to consider the positive definite quadratic form in s, $V(x) = s^{\rm T}Qs$. A sliding mode exists on a submanifold of $s(x) = 0$ that lies in a region of the state space on which the time rate of change of V is negative. Assume that α is bounded by a continuous function, $|\alpha(x)| < \sigma_\alpha(x)$, similar to the bound on Δ. Upon differentiation of $V(x)$, it is easy to verify that the choice of control

$$u = -\sigma(x)\operatorname{sgn}(s^*(x)), \quad s^*(x) = \rho(x)QKz(x),$$

where

$$\sigma(x)|\rho(x)| > \bar{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x)$$

leads to

$$\dot{V} \leqslant 2(\bar{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x) - \sigma(x)|\rho(x)|) \|QKz(x)\|,$$

$\bar{\sigma}(KA)$ denotes the maximum singular value of KA. In this case it follows that $\dot{V}$ is negative wherever it is defined (everywhere but on the sliding manifold), so that the sliding manifold is indeed attractive.

3.2. Smooth approximation of VS controllers

Suppose that the switch is replaced by a smooth version of a switch. Specifically, $u = -\sigma(x)\,\mathrm{sgn}(s^*(x)) \to -\sigma(x)\tanh(s^*(x)/\varepsilon)$, $\varepsilon > 0$. Then $\dot{V}$ is not necessarily negative for s small. However, for any given $\delta > 0$ there exists a sufficiently small $\varepsilon > 0$ such that $\dot{V} < 0$, for $|s| > \delta$ and all trajectories enter the strip $|s(x)| < \delta$. We can establish more than that. Namely, the smoothed control steers the state into a neighborhood of $z = 0$, the size of which shrinks with the design (smoothing) parameter ε.

Proposition 3.2. *Consider the system* (3). *Assume*

(1) *a smooth bound on* α, $|\alpha(x)| < \sigma_\alpha(x)$,
(2) *a smooth bound on* Δ, $|\Delta(x,t)| < \sigma_\Delta(x)\ \forall t$,
(3) $K = [a_1\ a_2\ \dots\ a_{r-1}\ 1]$, *where the coefficients are chosen such that the following matrix is stable*

$$A_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_{r-1} \end{bmatrix},$$

(4) $u = -\sigma(x)\tanh(s^*(x)/\varepsilon)$, *where* $s^*(x) = \rho(x)QKz(x)$ *and* $\sigma(x) > \bar{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x)$.

Then for any $\delta > 0$ *there exists a sufficiently small* $\varepsilon > 0$ *such that all trajectories enter the ball* $\|z\| < \delta$ *in finite time and remains therein.*

Proof. Since $Kb = 1$, we can divide the state space into $\mathrm{Im}\, b \oplus \ker K$. Thus we define a transformation:

$$z = b\zeta_1 + N\zeta_2,$$

where the columns of N span $\ker K$. In these new coordinates the evolution equations are

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\alpha(x) + \Delta(x,t) - \rho(x)\sigma(x)\tanh(s^*(x)/\varepsilon)).$$

In addition, $s = Kz = \zeta_1$. Furthermore, $\mathrm{Re}\,\lambda(A_4) < 0$ by design ($A_4 \sim A_s$). Hence, there exists matrices, $Q_0 \geqslant 0, R \geqslant 0$ such that

(1) $z^{\mathrm{T}}Q_0 z = 0$ for $z \in \mathrm{Im}\, b$ and $z^{\mathrm{T}}Q_0 z > 0$ otherwise.
(2) $d(z^{\mathrm{T}}Q_0 z)/\mathrm{d}t = -z^{\mathrm{T}}Rz \leqslant -\lambda_{\min}\|\zeta_2\|^2$, where $\lambda_{\min}$ is the smallest nonzero eigenvalue of R.

Now, consider the Lyapunov function

$$V(z) = z^{\mathrm{T}}Q_0 z + (Kz)^{\mathrm{T}}QKz > 0, \quad \|z\| \neq 0$$

and compute

$$\dot{V} = 2\{Az\}^{\mathrm{T}}Q_0 z + 2[KAz + \alpha + \Delta]^{\mathrm{T}}QKz + 2u\rho QKz.$$

Now, we have

$$[KAz + \alpha + \Delta]^{\mathrm{T}}QKz + u\rho QKx \leqslant (\bar{\sigma}(KA)\|z\| + \sigma_\alpha + \sigma_\Delta)\|QKz\| - \sigma|\tanh(s^*/\varepsilon)|$$

and

$$2\{Az\}^{\mathrm{T}}Q_0 z \leqslant -\lambda_{\min}\|\zeta_2\|^2$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}V \leqslant -\lambda_{\min}\|\zeta_2\|^2 + 2[\hat{\sigma} - \sigma|\tanh(s^*/\varepsilon)|],$$

where

$$\hat{\sigma} = (\bar{\sigma}(KA)\|z(x)\| + \sigma_\alpha(x) + \sigma_\Delta(x))\|QKz(x)\|.$$

Thus since $\sigma > \hat{\sigma}$ by assumption, for any specified $\delta > 0$ there is an $\varepsilon > 0$ such that $\dot{V} \leqslant -c < 0$. Consequently, we have all trajectories entering the strip $\|s\| < \delta(\varepsilon)$ in finite time. Now, since $s = \zeta_1$, it follows that $\|s\| < \delta \Rightarrow \|\zeta_1\| < \delta$. Consequently, from the evolution equations and since A_4 is asymptotically (exponentially) stable we can conclude that all trajectories enter a ball with radius proportional to δ in finite time and remains therein. □

Remark 3.3. Notice that the internal dynamics (2) can be written $\dot{\xi} = F(\xi, 0) + \Delta F(\xi, z(t))$ with $\Delta F(\xi, 0) = 0$, both F and ΔF smooth. Suppose the zero dynamics $\dot{\xi}_0 = F(\xi_0, 0)$ are exponentially stable. In view of Proposition 3.2, we can choose ε to make δ as small as necessary to insure that (2) is ultimately bounded. If the zero dynamics are asymptotically stable but not exponentially stable, ultimate boundedness may not obtain. That is the difficulty in Example 3.1.

4. Backstepping design of VS controls

We will describe a backstepping procedure for SISO VS control system design in the presence of uncertain, possibly non-smooth, nonlinearities. The method differs from the usual backstepping techniques in the following ways: (1) the states are grouped in accordance with the appearance of the uncertainty in the system, and (2) the control designed at each step is a VS control. Consider a SISO nonlinear system in the (multi-state back-stepping) form [1]

$$x_i^{(n_i)} = x_{i+1} + \Delta_i(x,t), \quad i = 1,\ldots,p-1,$$

$$x_p^{(n_p)} = \alpha(x) + \rho(x)u + \Delta_p(x,t), \tag{4}$$

$$y = x_1.$$

We assume that the (possibly non-smooth) uncertainties $\Delta_i(x,t)$ are bounded by smooth, non-negative functions $\sigma_i(x)$, i.e.,

$$0 \leqslant |\Delta_i(x,t)| \leqslant \sigma_i(x) \quad \forall t. \tag{5}$$

Such a model might arise by reduction of a smooth nominal system to regular from and applying the transformation to the uncertain system. The basic idea is very simple. At each of $p-1$ stages we design a 'pseudo-control' v_k, at the kth step, using the system (with $v_0 = 0$)

$$x_i^{(n_i)} = v_i + \Delta_i(x,t), \quad i = 1,\ldots,k < p,$$

$$y_k = x_k - v_{k-1}(x_1,\ldots,x_{(k-1)}^{n_{k-1}}).$$

and at the last (pth) stage we design the actual control, u, using the system

$$x_i^{(n_i)} = v_i + \Delta_i(x,t), \quad i = 1,\ldots,p-1,$$

$$x_p^{(n_p)} = \alpha(x) + \rho(x)u + \Delta_p(x,t).$$

Let us define the procedure in detail.

Algorithm 1 (VS Backstepping Algorithm). *The state transformation and control are constructed sequentially as follows*:

(1) $k = 1$. *Define the vector fields* $\hat f_1$, g_1 *and the scalar function* $\hat h_1$:

$$\hat f_1 = \begin{bmatrix} \dot x_1 \\ \vdots \\ x_1^{(n_1-1)} \\ 0 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$y_1 = \hat h_1(x_1) = x_1.$$

Now define the new state variables:

$$z_j^1 = y_1^{(j-1)} = L_{\hat f_1}^{j-1} \hat h_1, \quad j = 1,\ldots,n_1,$$

which leads to the state space description

$$\dot Z^1 = f_1(Z^1) + g_1 v_1 = \begin{bmatrix} z_2^1 \\ \vdots \\ z_{n_1}^1 \\ L_{\hat f_1}^{n_1} \hat h_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v_1,$$

$$y_1 = h_1(Z^1) = z_1^1,$$

where $Z^1 = [z_1^1,\ldots,z_{n_1}^1]^{\mathrm T}$. *Now, design the smoothed variable structure controller* v_1.

(2) $k = 2,\ldots,p-1$. *Define* $\hat f_k$, g_k, *and* $\hat h_k$

$$\hat f_k = \begin{bmatrix} f_{k-1}(Z^{k-1}) + g_{k-1} v_{k-2}(z^{k-2}) \\ \dot x_k \\ \vdots \\ x_k^{(n_k-1)} \\ 0 \end{bmatrix},$$

$$g_k = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$y_k = \hat h_k(Z^{k-1}, x_k) = x_k - v_{k-1}(z^{k-1}).$$

Define the next group of new states

$$z_j^k = y_k^{(j-1)} = L_{\hat f_k}^{j-1} \hat h_k, \quad j = 1,\ldots,n_k.$$

Write the state space equations in terms of the new states: $\dot Z^k = f_k(Z^k) + g_k v_k$,

$$f_k(Z^k) = \begin{bmatrix} f_{k-1}(Z^{k-1}) + g_{k-1} v_{k-2}(z^{k-2}) \\ z_2^k \\ \vdots \\ z_{n_k}^k \\ L_{\hat f}^{n_k} \hat h_k \end{bmatrix},$$

$$y_k = h_k(Z^k) = z_1^k - v_{k-1}(z^{k-1}),$$

where

$$z^k = [z_1^k,\ldots,z_{n_k}^k]^{\mathrm T}, \quad Z^k = \begin{bmatrix} Z^{k-1} \\ z^k \end{bmatrix} \in R^{n_1} + \cdots + n_k$$

and design the smoothed variable structure control v_k.

(3) $k = p$. $\hat f_p$, g_p, *and* $\hat h_p$ *are defined as above for general* k. *Now introduce the last group of new states*

$$z_j^p = y_p^{(j-1)} = L_{\hat f_p}^{j-1} \hat h_p, \quad j = 1,\ldots,n_p$$

to obtain the state space equations.

$$\dot Z^p = f_p(Z^p) + g_p(\alpha + \rho v_p), \quad Z^p = \begin{bmatrix} Z^{p-1} \\ z^p \end{bmatrix},$$

$$y_p = h_p(Z^p) = z_1^p - v_{p-1}(z^{p-1}).$$

Finally, design the variable structure controller v_p.

[1] We leave out zero dynamics. If present, the same results obtain if they are exponentially stable.

Now we apply this transformation to the actual system (4).

Lemma 4.4. *Consider the transformation defined recursively according to Algorithm* (1). *When applied to the actual system* (4) *the transformed evolution equations are*

$$y_i^{(n_i)} = y_{i+1} + \alpha_i + \Delta_i + v_i(y_i, \ldots, y_i^{(n_i-1)}),$$
$$i = 1, \ldots, p-1,$$
$$y_p^{(n_p)} = \alpha + \alpha_p + \Delta_p + \rho u(y_p, \ldots y_p^{(n_p-1)}). \quad (6)$$

Proof. Notice that at each stage of Algorithm (1), for $k = 1, \ldots, p-1$, n_k new state variables are defined and n_k first-order equations are added to the system. The first $n_k - 1$ equations come from the state definitions, i.e., the defining equations

$$z_j^k = y_k^{(j-1)} = L_{\hat{f}_k}^{j-1} \hat{h}_k, \quad j = 1, \ldots, n_k$$

imply

$$\dot{y}_k = \dot{z}_1^k = z_2^k, \ldots, y_k^{(n_k-1)} = \dot{z}_{n_k-1}^k = z_{n_k}^k.$$

The final equation is obtained by differentiating the last definition and using the evolution equation $x_k^{(n_k)} = v_k$ in the nominal case and $x_k^{(n_k)} = \Delta_k + v_k$ in the actual case, leading to

$$y_k^{(n_k)} = \dot{z}_{n_k}^k = L_{\hat{f}_k}^{n_k} \hat{h}_k + L_{\hat{g}_k} L_{\hat{f}_k}^{n_k-1} \hat{h}_k v_k = L_{\hat{f}_k}^{n_k} \hat{h}_k + v_k$$

in the nominal case, and

$$y_k^{(n_k)} = \dot{z}_{n_k}^k = L_{\hat{f}_k}^{n_k} \hat{h}_k + \Delta_k + v_k$$

in the actual case. The case $k = p$ is similar except that $\alpha + \rho v_p$ is replaced by $\alpha + \Delta_p + \rho v_p$. □

Remark 4.5. In the above result, α and Δ_i, $i = 1, \ldots, p$ are explicit functions of the original states and time, i.e., $\alpha = \alpha(x)$, and $\Delta_i = \Delta_i(x, t)$, $i = 1, \ldots, p$.

Stability is established in the following proposition.

Proposition 4.6. *Consider the system* (4) *and suppose the uncertainties Δ_i satisfy the inequality* (5) *with continuous bounding functions σ_i, and α also has a continuous bounding function σ_α. Suppose that a controller is designed via the backstepping procedure of Algorithm* (1) *and each control v_k, $k = 1, \ldots, p$ is a smoothed variable structure controller designed in accordance with the assumptions of Proposition* (3.2). *Then for any given $\delta > 0$ there is a sufficiently small smoothing parameter $\varepsilon > 0$ such that all trajectories enter the ball $\|y\| < \delta$.*

Proof. The pth system

$$y_p^{(n_p)} = \alpha + \alpha_p + \Delta_p + \rho v_p(y_p, \ldots, y_p^{(n_p-1)}) \quad (7)$$

satisfies the conditions of Proposition (3.2) with $z_i = y_p^{(i-1)}$, $i = 1, \ldots, n_p$. Hence, we conclude that y_p (and its $n_p - 1$ derivatives) will be driven, in finite time, into a δ-neighborhood of the origin with a suitably small smoothing parameter. Now, the $p-1$ system is

$$y_{p-1}^{(n_{p-1})} = y_p(t) + \alpha_{p-1} + \Delta_{p-1} + v_{p-1}(y_{p-1}, \ldots y_{p-1}^{(n_{p-1}-1)}) \quad (8)$$

and $|y_p(t)| \leqslant \delta$, $\forall t > t^* < \infty$. Thus, we can incorporate $y_p(t)$ into $\Delta_{p-1}(x, t)$. It follows that (8) satisfies the conditions of Proposition (3.2) for $t > t^*$, $z_i = y_{p-1}^{(i-1)}$, $i = 1, \ldots, n_{p-1}$, so that y_{p-1} (and its $n_{p-1} - 1$ derivatives) will be driven, in finite time, into a δ-neighborhood of the origin with a suitably small smoothing parameter. We continue in this way for systems $k = p-2, \ldots, 1$ to establish the conclusion of the theorem. □

5. Example

Consider the two degrees of freedom, fourth-order motor-load drive system illustrated in Fig. 1. It involves friction at two different locations and is representative of systems of practical interest. The friction functions are defined by Eq. (9). The calculations that follow have been carried out using *Mathematica*.

$$\varphi_1(\omega_1) = -\tfrac{1}{2}\,\omega_1 - \tfrac{1}{10}(1 + \tfrac{1}{10}\,\mathrm{e}^{-(50\omega_1)^2})\,\mathrm{sgn}\,\omega_1,$$
$$\varphi_2(\omega_2) = -\tfrac{1}{2}\,\omega_2 - \tfrac{1}{10}\,\mathrm{sgn}\,\omega_2. \quad (9)$$

We begin by reducing the nominal system to normal form. The required transformation is

$$\begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ (20x_1 + x_2 + 2x_3)/20 \\ (20x_2 + x_3 + 2x_4)/20 \end{bmatrix}.$$

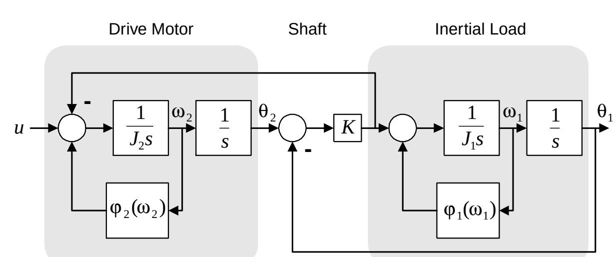

Fig. 1. A typical drive system consisting of a motor and an inertial load. The nonlinear friction functions, φ_1 and φ_2, contain uncertain discontinuous components.

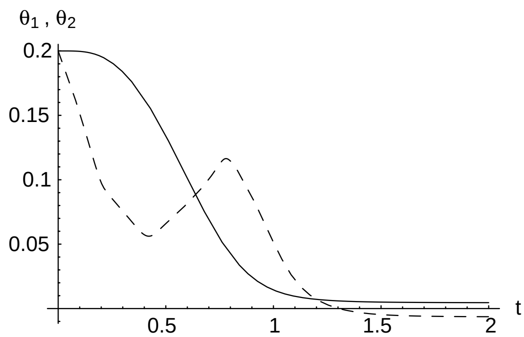

Fig. 2. The closed loop transient response is illustrated for the system intially at rest with the shaft offset by 0.2 rad. Load angle θ_1 is the solid line and motor angle θ_2 is dashed.

When the transformation is applied to the actual (perturbed) system, we obtain

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3 - (0.1 + 0.01\mathrm{e}^{-2500x_2^2})\,\mathrm{sgn}(x_2),$$

$$\dot{x}_3 = x_4 + 0.005(10 + \mathrm{e}^{-2500x_2^2})\,\mathrm{sgn}(x_2),$$

$$\dot{x}_4 = \tfrac{39}{400}(10 + \mathrm{e}^{-2500x_2^2})\,\mathrm{sgn}(x_2) - \tfrac{1}{4}(40x_2 + 81x_3 + 4x_4 + 4\,\mathrm{sgn}(20x_2 + x_3 + 2x_4)) + 10u.$$

The backstepping procedure requires three steps, because uncertainties enter the right-hand sides of the second, third and fourth equations. The resulting controller is

$$u = -(5 + 1250\,\mathrm{Abs}[x_2] + 500\,\mathrm{Abs}[x_3] + 50\,\mathrm{Abs}[x_4]) \times \tanh[250(x_4 + 5\tanh[10(x_3 + \tanh \times [24.1x_1 + 10x_2])])].$$

Numerous simulations have been run. A typical result is shown in Fig. 2. From these initial conditions the ultimate error appears quite small, but it is not zero. By decreasing the smoothing parameter the error is reduced. On the other hand the (peak) control effort increases. The switching control bounds were selected in accordance with estimates of bounds required by Proposition (4.6). Experiments show that these bounds (and hence the control peaks) cannot be substantially reduced. Our example exaggerates the size of the nondifferentiable, uncertain friction component and the shaft is more flexible than in the applications of interest to us. These factors make the control problem more difficult and increase the required control magnitudes.

6. Conclusions

In this paper, we have introduced a new method for design of control systems for a class of SISO systems with nondifferentiable, uncertain nonlinearities such as friction. The resulting feedback control is a smoothed, variable structure controller designed using a multi-state backstepping procedure. In preliminary studies the controller appears to be effective in dealing with the difficult problem of friction sandwiched between dynamical elements. Very little needs to be known about the details of the friction model. Only bounds on the friction function are required. This is especially important when friction depends on position, varies with time, or is otherwise difficult to characterize. The intended application is to situations where uncertain friction forces are relatively small, but nonetheless significant by virtue of the required precision.

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